# Numerical Solution of Fuzzy Differential Equation by Runge-Kutta Method

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## Abstract

In this paper, the numerical algorithms for solving 'fuzzy ordinary differential equations' are considered. A scheme based on the $4^{th}$ order Runge-Kutta method is discussed in detail and it is followed by a complete error analysis. The algorithm is illustrated by solving some linear and nonlinear fuzzy Cauchy problems.

## 1. Introduction

The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [1]. It was followed up by D.Dubois, H. Prade in [2], who defined and used the extension principle. The fuzzy differential equation and the initial value problem were regularly treated by O. Kaleva in [5] and [6], by S. Seikkala in [7]. The numerical method for solving fuzzy differential equations is introduced by M. Ma, M. Friedman, A. Kandel in [10] by the standard Euler method. The structure of this paper organizes as follows:

In section 2 some basic results on fuzzy numbers and definition of a fuzzy derivative, which have been discussed by S. Seikkala in [10], are given. In section 3 we define the problem, this is a fuzzy Cauchy problem whose numerical solution is the main interest of this work. The numerically solving fuzzy differential equation by $4^{th}$ order Runge-Kutta method is discussed in section 4. The proposed algorithm is illustrated by solving some examples in section 5 and conclusion is in section 6.

**Keywords:** Fuzzy Differential Equation, $4^{th}$ Order Runge-Kutta Method, Fuzzy Cauchy Problem.

## 2. Preliminaries

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); \quad a \le t \le b, \\ y(a) = \alpha. \end{cases} \tag{1}$$

The basis of all Rung-Kutta methods is to express the difference between the value of $y$ at $t_{n+1}$ and $t_n$ as

$$y_{n+1} - y_n = h\sum_{i=1}^{m} w_i k_i. \tag{2}$$

where the $w_i$ ' s for $i = 1,2,...m,$ are constants and

$$k_i = f(t_n + \alpha_i h, y_n + h\sum_{j=1}^{i-1} \beta_{ij} k_j). \tag{3}$$

Equation (2) is to be coincident whit Taylor series order $m.$ Therefore, the truncation error $T_m$, can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}).$$

The true magnitude of $\gamma_m$ will generally be much less than the bound of Theorem 2.1. Thus, if the $O(h^{m+2})$ term is small compared with $\gamma_m h^{m+1},$ as we expect, to be so if $h$ is small enough, then the bound on $\gamma_m h^{m+1},$ will usually be a bound on the error as a whole. The famous nonzero constants $\alpha_i, \beta_{ij}$ in 4th order Runge-Kutta method are

$$\alpha_1 = 0\,, \alpha_2 = \alpha_3 = \tfrac{1}{2}\,, \alpha_4 = 1\,, \beta_{21} = \tfrac{1}{2}\,, \beta_{32} = \tfrac{1}{2}\,, \beta_{43} = 1,$$

and we have, see [9]

$$\begin{aligned} & y_0 = \alpha\,, \\ & k_1 = f(t_i, y_i)\,, \\ & k_2 = f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} k_1)\,, \\ & k_3 = f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} k_2)\,, \\ & k_4 = f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} k_3)\,, \\ & y_{i+1} = y_i + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{aligned} \tag{4}$$

where

$$a = t_0 < t_1 < ... < t_N = b \text{ and } h = \frac{b-a}{N} = t_{i+1} - t_i\,. \tag{5}$$

**Theorem 2.1.** Let $f(t,y)$ belong to $C^4[a,b]$ and let its partial derivatives be bounded and assume there exist, $P,M$ , positive numbers, such that

$$|f(t,y)|<M,\ |\frac{\partial^{i+j} f}{\partial t^i \partial y^j}|<\frac{P^{i+j}}{M^{j-1}},\quad i+j\le m,$$

then in the 4th order Rung-Kutta method (Proof [9]).

$$y(t_{i+1})-y_{i+1}=\tfrac{73}{720}h^5MP^4+O(h^6).$$

A triangular fuzzy number $\nu$, is defined by three numbers $a_1<a_2<a_3$ where the graph of $\nu(x)$, the membership function of the fuzzy number $\nu$, is a triangle with base on the interval $[a_1,a_3]$, and vertex at $x=a_2$. We specify $\nu$ as $(a_1/a_2/a_3)$. We will write: $(1)\nu>0$ if $a_1>0$; $(2)\nu\ge 0$ if $a_1\ge 0$; $(3)\nu<0$ if $a_3<0$; and $(4)\nu\le 0$ if $a_3\le 0$

Let $E$ be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level sets It means that if $v\in E$ then the r-level set

$$[v]_r=\{s\,|\,v(s)\ge r\},\quad 0<r\le 1,$$

is a closed bounded interval which is denoted by

$$[v]_r=[v_1(r),v_2(r)].$$

Let $I$ be a real interval. A mapping $x:I\to E$ is called a fuzzy process and its r-level set is denoted by

$$[x(t)]_r=[x_1(t;r),x_2(t;r)],\quad t\in I,\quad r\in(0,1].$$

The derivative $x'(t)$ of a fuzzy process $x$ is defined by

$$[x'(t)]_r=[x'_1(t;r),x'_2(t;r)],\quad t\in I,\quad r\in(0,1],$$

provided that this equation defines a fuzzy number, as in Sikkala [7].

**Lemma 2.1.** Let $v,w\in E$ and $s$ be a scalar (.see [7]). Then for $r\in(0,1]$

$$\begin{aligned}
[v+w]_r &= [v_1(r)+w_1(r),v_2(r)+w_2(r)],\\
[v-w]_r &= [v_1(r)-w_2(r),v_2(r)-w_1(r)],\\
[v.w]_r &= [\min\{v_1(r).w_1(r),v_1(r).w_2(r),v_2(r).w_1(r),v_2(r).w_2(r)\},\\
&\qquad \max\{v_1(r).w_1(r),v_1(r).w_2(r),v_2(r).w_1(r),v_2(r).w_2(r)\}],\\
[s.v]_r &= s[v]_r.
\end{aligned}$$

## 3. A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), \ t \in I = [0, T], \\ y(0) = y_0, \end{cases} \qquad (6)$$

where $f$ is a continuous mapping from $R_+ \times R$ into $R$ and $y_0 \in E$ with r-level sets

$$[y_0]_r = [y_1(0; r), y_2(0; r)], \quad r \in (0,1].$$

The extension principle of Zadeh leads to the following definition of $f(t, y)$ when $y = y(t)$ is a fuzzy number

$$f(t, y)(s) = \sup\{y(\tau) \mid s = f(t, \tau)\}, \quad s \in R.$$

It follows that

$$[f(t, y)]_r = [f_1(t, y; r), f_2(t, y; r)], \quad r \in (0,1],$$

where

$$\begin{aligned} f_1(t, y; r) &= \min\{ f(t, u) \mid u \in [y_1(t; r), yx_2(t; r)]\}, \\ f_2(t, y; r) &= \max\{ f(t, u) \mid u \in [y_1(t; r), y_2(t; r)]\}. \end{aligned} \qquad (7)$$

**Theorem 3.1.** *Let $f$ satisfy*

$$| f(t, v) - f(t, \bar{v}) | \le g(t, | v - \bar{v}|), \quad t \ge 0, \quad v, \bar{v} \in R,$$

*where $g : R_+ \times R_+ \to R_+$ is a continuous mapping such that $r \to g(t, r)$ is nondecreasing, the initial value problem*

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \qquad (8)$$

*has a solution on $R_+$ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (8) for $u_0 = 0$. Then the fuzzy initial value problem (6) has a unique fuzzy solution.*

Proof [7].

## 4. 4th Order Runge-Kutta Method

Let the exact solution $[Y(t)]_r = [Y_1(t; r), Y_2(t; r)]$ be approximated by some $[y(t)]_r = [y_1(t; r), y_2(t; r)]$. From (2),(3) we define

$$
\begin{aligned}
y_1(t_{n+1};r) - y_1(t_n;r) &= h\sum_{i=1}^{4} w_i k_{i,1}(t_n, y(t_n;r)),\\
y_2(t_{n+1};r) - y_2(t_n;r) &= h\sum_{i=1}^{4} w_i k_{i,2}(t_n, y(t_n;r)),
\end{aligned} \tag{9}
$$

where the $w_i$'s are constants and

$$
\begin{aligned}
&[k_i(t, y(t;r))]_r = [k_{i,1}(t, y(t;r)), k_{i,2}(t, y(t;r))],\ i = 1,2,3,4\\
&k_{i,1}(t_n, y(t_n;r)) = f(t_n + \alpha_i h\ ,\ y_1(t_n) + h\sum_{j=1}^{i-1}\beta_{i,j} k_{j,1}(t_n, y(t_n;r))),\\
&k_{i,2}(t_n, y(t_n;r)) = f(t_n + \alpha_i h\ ,\ y_2(t_n) + h\sum_{j=1}^{i-1}\beta_{i,j} k_{j,2}(t_n, y(t_n;r)))
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
k_{1,1}(t, y(t;r)) &= \min\{ f(t,u) \mid u \in [y_1(t;r), y_2(t;r)]\},\\
k_{1,2}(t, y(t;r)) &= \max\{ f(t,u) \mid u \in [y_1(t;r), y_2(t;r)]\},\\
k_{2,1}(t, y(t;r)) &= \min\{ f(t + \tfrac{h}{2}, u) \mid u \in [z_{1,1}(t; y(t;r)), z_{1,2}(t; y(t;r))]\},\\
k_{2,2}(t, y(t;r)) &= \max\{ f(t + \tfrac{h}{2}, u) \mid u \in [z_{1,1}(t; y(t;r)), z_{1,2}(t; y(t;r))]\},\\
k_{3,1}(t, y(t;r)) &= \min\{ f(t + \tfrac{h}{2}, u) \mid u \in [z_{2,1}(t; y(t;r)), z_{2,2}(t; y(t;r))]\},\\
k_{3,2}(t, y(t;r)) &= \max\{ f(t + \tfrac{h}{2}, u) \mid u \in [z_{2,1}(t; y(t;r)), z_{2,2}(t; y(t;r))]\},\\
k_{4,1}(t, y(t;r)) &= \min\{ f(t + h, u) \mid u \in [z_{3,1}(t; y(t;r)), z_{3,2}(t; y(t;r))]\},\\
k_{4,2}(t, y(t;r)) &= \max\{ f(t + h, u) \mid u \in [z_{3,1}(t; y(t;r)), z_{3,2}(t; y(t;r))]\}.
\end{aligned} \tag{11}
$$

Where in the 4$^{\text{th}}$ order Runge-Kutta method,

$$
\begin{aligned}
z_{1,1}(t, y(t;r)) &= y_1(t;r) + \tfrac{h}{2} k_{1,1}(t, y(t;r)) \quad , \quad z_{1,2}(t, y(t;r)) = y_2(t;r) + \tfrac{h}{2} k_{1,2}(t, y(t;r)),\\
z_{2,1}(t, y(t;r)) &= y_1(t;r) + \tfrac{h}{2} k_{2,1}(t, y(t;r)) \quad , \quad z_{2,2}(t, y(t;r)) = y_2(t;r) + \tfrac{h}{2} k_{2,2}(t, y(t;r)),\\
z_{3,1}(t, y(t;r)) &= y_1(t;r) + h k_{3,1}(t, y(t;r)) \quad , \quad z_{3,2}(t, y(t;r)) = y_1(t;r) + h k_{3,2}(t, y(t;r)).
\end{aligned} \tag{12}
$$

Define,

$$
\begin{aligned}
F[t, y(t;r)] &= k_{1,1}(t, y(t;r)) + 2k_{2,1}(t, y(t;r)) + 2k_{3,1}(t, y(t;r)) + k_{4,1}(t, y(t;r)),\\
G[t, y(t;r)] &= k_{1,2}(t, y(t;r)) + 2k_{2,2}(t, y(t;r)) + 2k_{3,2}(t, y(t;r)) + k_{4,2}(t, y(t;r)).
\end{aligned} \tag{13}
$$

The exact and approximate solutions at $t_n$ , $0 \le n \le N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)]$ and $[y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$, respectively. The solution is calculated at grid points of (5). By (9),(13) we have

$$
\begin{aligned}
Y_1(t_{n+1};r) &\approx Y_1(t_n;r) + \tfrac{h}{6} F[t_n, Y(t_n);r],\\
Y_2(t_{n+1};r) &\approx Y_2(t_n;r) + \tfrac{h}{6} G[t_n, Y(t_n);r].
\end{aligned} \tag{14}
$$

We define

$$y_1(t_{n+1};r) = y_1(t_n;r) + \tfrac{h}{6} F[t_n, y(t_n);r],$$
$$y_2(t_{n+1};r) = y_2(t_n;r) + \tfrac{h}{6} G[t_n, y(t_n);r], \tag{15}$$

the following lemmas will be applied to show convergence of these approximates,i.e.,

$$\lim_{h\to 0} y_1(t;r) = Y_1(t;r),$$
$$\lim_{h\to 0} y_2(t;r) = Y_2(t;r).$$

**Lemma 4.1.** *Le tthe sequence of numbers* $\{W_n\}_{n=0}^{N}$ *satisfy*

$$|W_{n+1}| \le A|W_n| + B, \quad 0 \le n \le N-1,$$

*for some given positive constants* $A$ *and* $B$ *(* Proof [10]).*Then*

$$|W_n| \le A^n |W_0| + B\frac{A^n - 1}{A-1}, \quad 0 \le n \le N .$$

**Lemma 4.2.** Let the sequence of numbers $\{W_n\}_{n=0}^{N}$ , $\{V_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le |W_n| + A.\max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \le |V_n| + A.\max\{|W_n|, |V_n|\} + B,$$

*for some given positive constants be* $A$ *and* $B$ *., and denote*

$$|U_n| = |W_n| + |V_n|, \quad 0 \le n \le N .$$

*Then*

$$|U_n| \le \bar{A}^n |U_0| + \bar{B}\frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \le n \le N ,$$

*where* $\bar{A} = 1 + 2A$ *and* $\bar{B} = 2B$ *(* Proof [10]).

Let $F(t,u,v)$ and $G(t,u,v)$ be obtained by substituting $[y(t)]_r = [u,v]$ in (13).

$$F[t,u,v] = k_{1,1}(t,u,v) + 2k_{2,1}(t,u,v) + 2k_{3,1}(t,u,v) + k_{4,1}(t,u,v),$$
$$G[t,u,v] = k_{1,2}(t,u,v) + 2k_{2,2}(t,u,v) + 2k_{3,2}(t,u,v) + k_{4,2}(t,u,v).$$

The domain where $F$ and $G$ are defined is therefore

$$K = \{(t,u,v) \mid 0 \le t \le T, -\infty < v < \infty, -\infty < u \le v\} .$$

**Theorem 4.1.** *Let $F(t,u,v)$ and $G(t,u,v)$ belonged to $C^4(K)$ and let the partial derivatives of $F$ and $G$ be bounded over $K$. Then, for arbitrary fixed $r: 0 \le r \le 1$, the approximative solutions (14) converge to the exact solutions $Y_1(t;r)$ and $Y_2(t;r)$ uniformly in t.*

**Proof.** It is sufficient to show

$$\lim_{h\to 0} y_1(t_N;r) = Y_1(t_N;r),$$

$$\lim_{h\to 0} y_2(t_N;r) = Y_2(t_N;r),$$

where $t_N = T$. For $n = 0,1,...,N-1$, by using Taylor theorem we get

$$\begin{aligned} Y_1(t_{n+1};r) &= Y_1(t_n;r) + \tfrac{h}{6} F[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{73}{720} h^5 MP^4 + O(h^6), \\ Y_2(t_{n+1};r) &= Y_2(t_n;r) + \tfrac{h}{6} G[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{73}{720} h^5 MP^4 + O(h^6), \end{aligned} \tag{16}$$

denote

$$W_n = Y_1(t_n;r) - y_1(t_n;r),$$
$$V_n = Y_2(t_n;r) - y_2(t_n;r).$$

Hence from (15) and (16)

$$W_{n+1} = W_n + \tfrac{h}{6}\{F[t_n, Y_1(t_n;r), Y_2(t_n;r)] - F[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \tfrac{73}{720} h^5 MP^4 + O(h^6),$$

$$V_{n+1} = V_n + \tfrac{h}{6}\{G[t_n, Y_1(t_n;r), Y_2(t_n;r)] - G[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \tfrac{73}{720} h^5 MP^4 + O(h^6).$$

Then

$$|W_{n+1}| \le |W_n| + \tfrac{1}{3} Lh.\max\{|W_n|, |V_n|\} + \tfrac{73}{720} h^5 MP^4 + O(h^6),$$

$$|V_{n+1}| \le |V_n| + \tfrac{1}{3} Lh.\max\{|W_n|, |V_n|\} + \tfrac{73}{720} h^5 MP^4 + O(h^6),$$

for $t \in [0,T]$ and $L > 0$ is a bound for the partial derivatives of $F$ and G. Thus by

lemma 4.2

$$|W_n| \le (1+\tfrac{2}{3}Lh)^n |U_0| + (\tfrac{73}{360} h^5 MP^4 + O(h^6)) \frac{(1+\frac{2}{3}Lh)^n - 1}{\frac{2}{3}Lh},$$

$$|V_n| \le (1+\tfrac{2}{3}Lh)^n |U_0| + (\tfrac{73}{360} h^5 MP^4 + O(h^6)) \frac{(1+\frac{2}{3}Lh)^n - 1}{\frac{2}{3}Lh},$$

where $|U_0|=|W_0|+|V_0|$. In particular

$$|W_N|\le(1+\tfrac{2}{3}Lh)^N|U_0|+(\tfrac{73}{240}h^4MP^4+O(h^5))\frac{(1+\tfrac{2}{3}Lh)^{\frac{T}{h}}-1}{L},$$
$$|V_N|\le(1+\tfrac{2}{3}Lh)^N|U_0|+(\tfrac{73}{240}h^4MP^4+O(h^5))\frac{(1+\tfrac{2}{3}Lh)^{\frac{T}{h}}-1}{L}.$$

Since $W_0=V_0=0$ we obtain

$$|W_N|\le(\tfrac{73}{240}MP^4)\frac{e^{\frac{2}{3}LT}-1}{L}h^4+O(h^5),$$
$$|V_N|\le(\tfrac{73}{240}MP^4)\frac{e^{\frac{2}{3}LT}-1}{L}h^4+O(h^5)$$

and if $h\to 0$ we get $W_N\to 0$ and $V_N\to 0$ which completes the proof.

## 5. Examples

**Example 5.1.** *Consider the fuzzy initial value problem, [10],*

$$\begin{cases} y'(t)=y(t),\ t\in I=[0,T], \\ y(0)=(0.75+0.25r,1.125-0.125r),\ 0<r\le 1. \end{cases}$$

By using 4th order Runge-Kutta method we have

$$y_1(t_{n+1};r)=y_1(t_n;r)[1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}],$$
$$y_2(t_{n+1};r)=y_2(t_n;r)[1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}],$$

The exact solution is given by

$$Y_1(t;r)=y_1(0;r)e^t \quad , \quad Y_2(t;r)=y_2(0;r)e^t$$

which at $t=1$,

$$Y(1;r)=[(0.75+0.25r)\,e,(1.125-0.125r)\,e],\ 0<r\le 1.$$

The exact and approximate solutions are compared and plotted at $t=1$ in Figure (5.1).

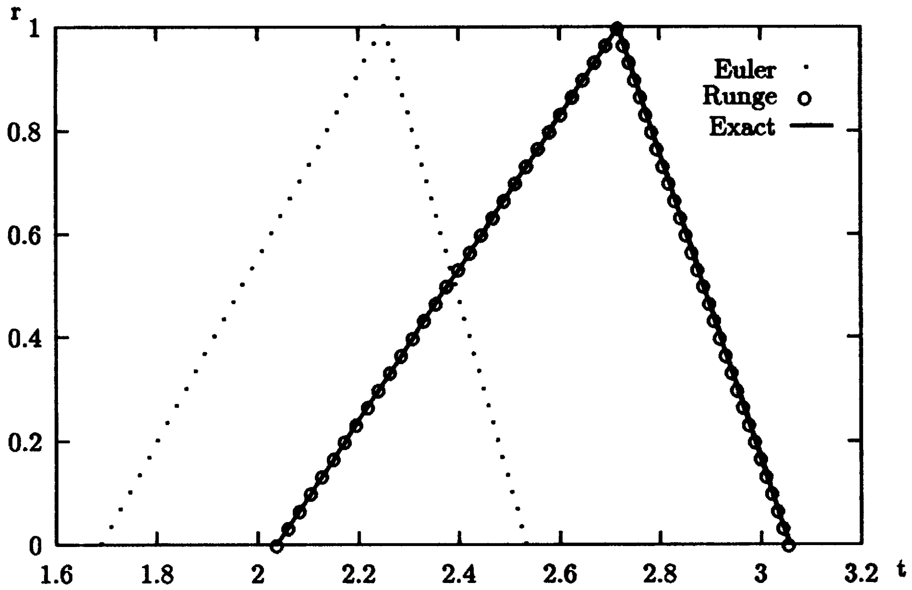


**Figure (5.1), (** $h=0.5$ **)**

**Example 5.2.** *Consider the fuzzy initial value problem, [10],*

$$\begin{cases} y'(t)=ty(t),\ [a,b]=[-1,1], \\ y(-1)=(\sqrt{e}-0.5(1-r),\sqrt{e}+0.5(1-r)),\ 0<r\le 1. \end{cases}$$

We separate between two steps.

$(a)\ t<0$: The parametric form in this case is

$$y'_1(t;r)=t\,y_2(t;r) \quad , \quad y'_2(t;r)=t\,y_1(t;r),$$

with the initial conditions given. The unique exact solution is

$$Y_1(t;r)=\frac{A-B}{2}y_2(0;r)+\frac{A+B}{2}y_1(0;r),$$
$$Y_2(t;r)=\frac{A+B}{2}y_2(0;r)+\frac{A-B}{2}y_1(0;r),$$

where $A=e^{\frac{(t^2-a^2)}{2}}$ , $B=\dfrac{1}{A}$ .

$(b)\ t\ge 0$: The parametric equations are

$$y'_1(t;r)=t\,y_1(t;r) \quad , \quad y'_2(t;r)=t\,y_2(t;r),$$

with the initial conditions given. The unique exact solution at $t > 0$ is

$$Y_1(t;r) = y_1(0;r)e^{\frac{t^2}{2}} \quad , \quad Y_2(t;r) = y_2(0;r)e^{\frac{t^2}{2}} .$$

By using 4th order Runge-Kutta method at $t_n, 0 \le n \le N$ we have

$$\begin{aligned}
k_{1,1}(t_n;r) &= \min\{t.u \mid u \in [y_1(t_n;r), y_2(t_n;r)]\},\\
k_{1,2}(t_n;r) &= \max\{t.u \mid u \in [y_1(t_n;r), y_2(t_n;r)]\},\\
k_{2,1}(t_n;r) &= \min\{(t+\tfrac{h}{2}).u \mid u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},\\
k_{2,2}(t_n;r) &= \max\{(t+\tfrac{h}{2}).u \mid u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},\\
k_{3,1}(t_n;r) &= \min\{(t+\tfrac{h}{2}).u \mid u \in [z_{2,1}(t_n;r), z_{2,2}(t_n;r)]\},\\
k_{3,2}(t_n;r) &= \max\{(t+\tfrac{h}{2}).u \mid u \in [z_{2,1}(t_n;r), z_{2,2}(t_n;r)]\},\\
k_{4,1}(t_n;r) &= \min\{(t+h).u \mid u \in [z_{3,1}(t_n;r), z_{3,2}(t_n;r)]\},\\
k_{4,2}(t_n;r) &= \max\{(t+h).u \mid u \in [z_{3,1}(t_n;r), z_{3,2}(t_n;r)]\}.
\end{aligned}$$

Where

$$\begin{aligned}
z_{1,1}(t_n;r) &= y_1(t_n;r) + \tfrac{h}{2}k_{1,1}(t_n;r) \quad , \quad z_{1,2}(t_n;r) = y_2(t_n;r) + \tfrac{h}{2}k_{1,2}(t_n;r),\\
z_{2,1}(t_n;r) &= y_1(t_n;r) + \tfrac{h}{2}k_{2,1}(t_n;r) \quad , \quad z_{2,2}(t_n;r) = y_2(t_n;r) + \tfrac{h}{2}k_{2,2}(t_n;r),\\
z_{3,1}(t_n;r) &= y_1(t_n;r) + hk_{3,1}(t_n;r) \quad , \quad z_{3,2}(t_n;r) = y_1(t_n;r) + hk_{3,2}(t_n;r).
\end{aligned}$$

By considering $t > 0$ and $t < 0$ ,the above minimizing and maximizing problems can be solved by GAMS software. The exact and approximate solutions are compared and plotted in Fig.(5.2).

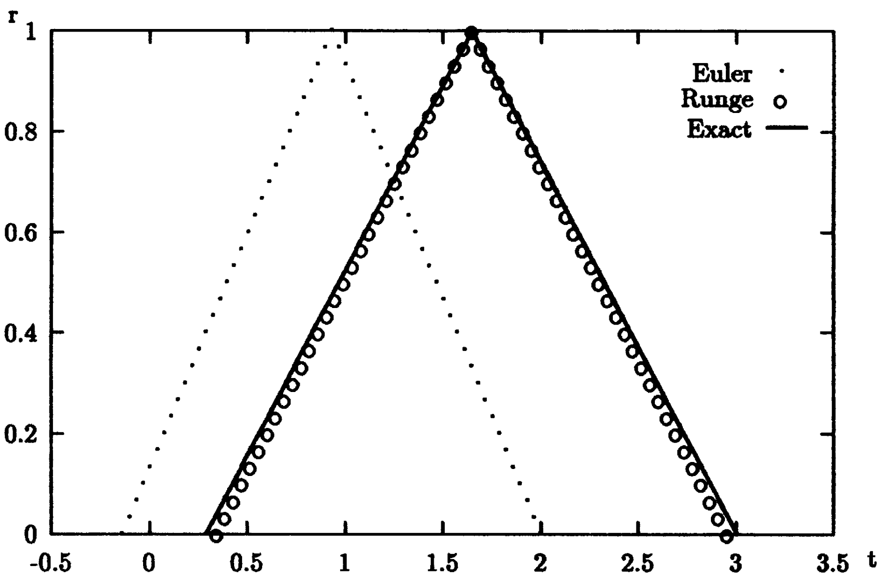


**Figure (5.2),** $(h = 0.4)$

**Example 5.3.** *Consider the fuzzy initial value problem*

$$y'(t) = c_1 y^2(t) + c_2, \quad y(0) = 0,$$

*where* $c_i > 0$, *for* $i = 1,2$ *are triangular fuzzy numbers, [11].*

The exact solution is given by

$$Y_1(t;r) = l_1(r)\tan(\omega_1(r)t),$$
$$Y_2(t;r) = l_2(r)\tan(\omega_2(r)t),$$

with

$$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)} \quad , \quad l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)},$$
$$\omega_1(r) = \sqrt{c_{1,1}(r)/c_{2,1}(r)} \quad , \quad \omega_2(r) = \sqrt{c_{1,2}(r)/c_{2,2}(r)},$$

where

$$[c_1]_r = [c_{1,1}(r), c_{1,2}(r)] \quad \text{and} \quad [k_2]_r = [c_{2,1}(r), c_{2,2}(r)]$$

and

$$c_{1,1}(r) = 0.5 + 0.5r\ ,\ c_{1,2}(r) = 1.5 - 0.5r\ ,\ c_{2,1}(r) = 0.75 + 0.25r\ ,\ c_{2,2}(r) = 1.25 - 0.25r.$$

The r-level sets of $y'(t)$ are

$$Y'_1(t;r) = c_{2,1}(r)\sec^2(\omega_1(r)t),$$
$$Y'_2(t;r) = c_{2,2}(r)\sec^2(\omega_2(r)t),$$

which defines a fuzzy number. We have

$$f_1(t,y;r) = \min\{c_1u^2 + c_2 \mid u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$
$$f_2(t,y;r) = \max\{c_1u^2 + c_2 \mid u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

By using 4th order Runge-Kutta method at $t_n, 0 \le n \le N$ we have

$$k_{1,1}(t_n;r) = c_{1,1}(r).y_1^2(t_n;r) + c_{2,1}(r),$$
$$k_{1,2}(t_n;r) = c_{1,2}(r).y_2^2(t_n;r) + c_{2,2}(r),$$
$$k_{2,1}(t_n;r) = c_{1,1}(r).z_{1,1}^2(t_n;r) + c_{2,1}(r),$$
$$k_{2,2}(t_n;r) = c_{1,2}(r).z_{1,2}^2(t_n;r) + c_{2,2}(r),$$
$$k_{3,1}(t_n;r) = c_{1,1}(r).z_{2,1}^2(t_n;r) + c_{2,1}(r),$$
$$k_{3,2}(t_n;r) = c_{1,2}(r).z_{2,2}^2(t_n;r) + c_{2,2}(r),$$
$$k_{4,1}(t_n;r) = c_{1,1}(r).z_{3,1}^2(t_n;r) + c_{2,1}(r),$$
$$k_{4,2}(t_n;r) = c_{1,2}(r).z_{3,2}^2(t_n;r) + c_{2,2}(r).$$

where

$$z_{1,1}(t_n;r) = y_1(t_n;r) + \tfrac{h}{2}k_{1,1}(t_n;r) \quad , \quad z_{1,2}(t_n;r) = y_2(t_n;r) + \tfrac{h}{2}k_{1,2}(t_n;r),$$
$$z_{2,1}(t_n;r) = y_1(t_n;r) + \tfrac{h}{2}k_{2,1}(t_n;r) \quad , \quad z_{2,2}(t_n;r) = y_2(t_n;r) + \tfrac{h}{2}k_{2,2}(t_n;r),$$
$$z_{3,1}(t_n;r) = y_1(t_n;r) + hk_{3,1}(t_n;r) \quad , \quad z_{3,2}(t_n;r) = y_1(t_n;r) + hk_{3,2}(t_n;r).$$

There are two nonlinear programming problems which can be solved by GAMS software. Thus the suggested 4th order Runge-Kutta method of this paper can be used. The exact and approximate solutions are shown in Figure (5.3) at $t=1$.

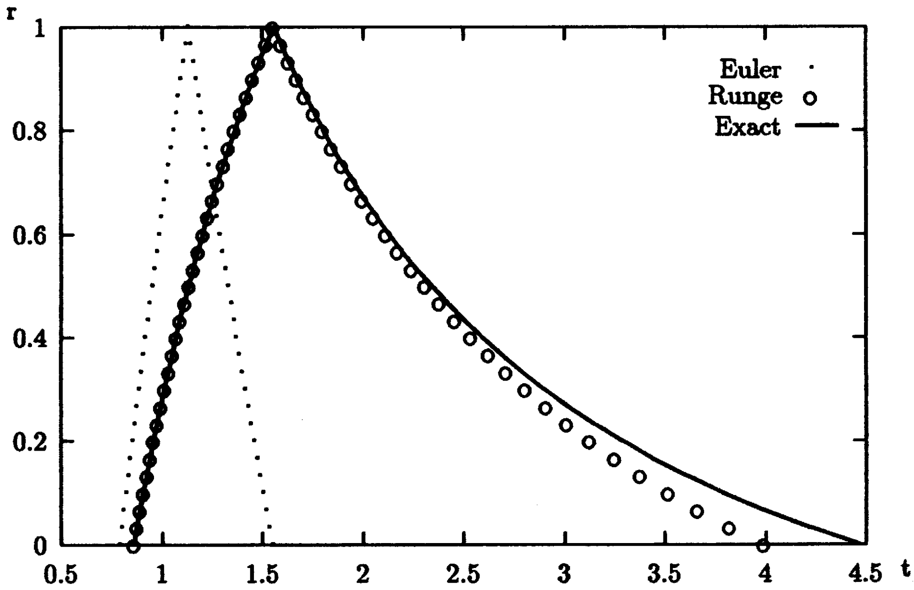


**Figure (5.3),** $(h = 0.5)$

## 6. Conclusion

We note that the convergence order of the Euler method in [10] is $O(h)$. It is shown that in proposed method, the convergence order is $O(h^4)$ and the comparison of solutions of examples (1) ,(2) of this paper and [10] shows that these solutions are better for these examples.

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