

# A certain $N$ -Generalized Principally Quasi-Baer Subring of the Matrix rings

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## Abstract

For a fixed positive integer  $n$ , we say a ring with identity is *n-generalized right principally quasi-Baer*, if for any principal right ideal  $I$  of  $R$ , the right annihilator of  $I^n$  is generated by an idempotent. This class of rings includes the right principally quasi-Baer rings and hence all prime rings. A certain  $n$ -generalized principally quasi-Baer subring of the matrix ring  $M_n(R)$  are studied, and connections to related classes of rings (e.g., p.q.-Baer rings and  $n$ -generalized p.p. rings) are considered<sup>1</sup>.

## 1. Introduction and Preliminaries

Throughout all rings are assumed to be associative with identity. From [12, 21], a ring  $R$  is (quasi-)Baer if the right annihilator of any (right ideal) nonempty subset of  $R$  is generated, as a right ideal, by an idempotent. Moreover, in [12] Clark proved the left-right symmetry of this condition. He uses this condition to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The class of quasi-Baer rings is a nontrivial generalization of the class of Baer rings. Every prime ring is quasi-Baer, hence prime rings with nonzero right singular ideal are quasi-Baer; but not Baer [24]. For a positive integer  $n > 1$ , the  $n \times n$  matrix ring over a non-Prüfer commutative domain is a prime quasi-Baer ring which is not a Baer ring by [27] and [21, p.17]. The  $n \times n$  ( $n > 1$ ) upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not

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Baer by an example due to Cohn; see [1], [20] and [5]. The theory of Baer and quasi-Baer rings has come to play an important role and major contributions have been made in recent years by a number of authors, including Birkenmeier, Chatters, Khuri, Kim, Hirano and Park (see, for example [1], [4-7], [16], [21], [26] and [28]).

A ring satisfying a generalization of *Rickart's condition* [30] (i.e., right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right p.p.-ring. A ring  $R$  is called a *right (resp. left) p.p.-ring* if every principal right (resp. left) ideal is projective.  $R$  is called a *p.p.-ring* (also called a Rickart ring [2, p.18]), if it is both right and left p.p.-ring. In [9] Chase shows the concept of p.p.-ring is not left-right symmetric. Small [30] shows that a right p.p.-ring  $R$  is Baer (so p.p), when  $R$  is orthogonally finite. Also it is shown by Endo [13] that a right p.p.-ring  $R$  is p.p when  $R$  is abelian (i.e., every idempotent is central). Finally Chatters and Xue [11] prove that in a duo (i.e., every one sided ideal is two sided) p.p.-ring  $R$ , if  $I$  is a finitely generated right projective ideal of  $R$ , then  $I$  is left projective and a direct summand of an invertible ideal. Following Birkenmeier et al. [7],  $R$  is called *right principally quasi-Baer* (or simply *right p.q.-Baer*), if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently,  $R$  is right p.q.-Baer if  $R$  modulo the right annihilator of any principal right ideal is projective. Similarly, left p.q.-Baer rings can be defined. If  $R$  is both right and left p.q.-Baer, then it is called p.q.-Baer. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, and all abelian p.p.-rings. A ring  $R$  is said to be *p-regular*, if for every  $x \in R$  there exists a natural number  $n$ , depending on  $x$ , such that  $x^n \in x^n R x^n$ . A ring  $R$  is called a *generalized right p.p.-ring* if for any  $x \in R$  the right ideal  $x^n R$  is projective for some positive integer  $n$ , depending on  $x$ , or equivalently, if for any  $x \in R$  the right annihilator of  $x^n$  is generated by an idempotent for some positive integer  $n$ , depending on  $x$ . A ring is called *generalized p.p.-ring*, if it is both generalized right and left p.p.-ring.

Note that Von Neumann regular rings are right (left) p.p.-rings by Goodearl [14,

Theorem 1.1], and a same argument as [14, Theorem 1.1] shows that  $\mathbf{p}$ -regular rings are generalized p.p.-rings. Right p.p.-rings are generalized right p.p. obviously. See [18] for more details.

**Definition 1.1.** Given a fixed positive integer  $n$ , we say a ring  $R$  is  $n$ -generalized right principally quasi Baer (or  $n$ -generalized right p.q.-Baer), if for all principal right ideal  $I$  of  $R$ , the right annihilator of  $I^n$  is generated by an idempotent. Left cases may be defined analogously.

The class of  $n$ -generalized right p.q.-Baer rings includes all right p.q.-Baer rings, (and hence all biregular rings, quasi-Baer rings, abelian p.p.-rings and *semicommutative* (i.e., if  $r(x)$  is an ideal for all  $x \in R$ ) generalized p.p. rings). Theorem 2.1 in section 2, allows us to construct examples of  $n$ -generalized p.q.-Baer rings that are not p.q.-Baer. Some conditions on the equivalence of  $n$ -generalized p.q.-Baer and  $n$ -generalized p.p.-rings are discussed. However, we show by examples that the class of  $n$ -generalized p.q.-Baer rings properly extends the aforementioned classes.

In this paper, we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. We study  $n$ -generalized p.q.-Baer subrings of the matrix ring  $M_n(R)$ . Theorem 2.2, enables us to generate examples of  $n$ -generalized p.q.-Baer subrings of the matrix ring  $M_n(R)$ . Theorem 2.2, which extends [18, Proposition 6], enables us to provide more examples of matrix rings, that are both  $n$ -generalized p.q.-Baer and  $n$ -generalized p.p.-ring. Connections to related classes of rings are investigated. Although the class of  $n$ -generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, and all abelian p.p. rings), however we show by examples that the class of  $n$ -generalized p.q.-Baer rings properly extends the aforementioned classes.

Note that, for a *reduced* ring (which has no nonzero nilpotent elements), we have  $l_R(Rx) = l_R((Rx)^n) = l_R(x^n) = l_R(x) = r_R(x) = r_R(x^n) = r_R((xR)^n) = r_R(xR)$ , for every  $x \in R$  and every positive integer  $n$ . Therefore reduced rings are semicommutative and semicommutative rings are abelian. Also for reduced rings the definitions of right p.q.-

Baer,  $n$ -generalized right p.q.-Baer, generalized p.p. and p.p.-ring are coincide. This leads one ask whether commutative reduced rings are  $n$ -generalized p.q.-Baer. However, the answer is negative by the following.

**Example 1.2.** Let  $p$  be a prime number and  $R = \{(a,b) \in Z \oplus Z \mid a \equiv b \pmod{p}\}$ , then  $R$  is a commutative reduced ring. Note that the only idempotents of  $R$  are  $(0,0)$  and  $(1,1)$ . One can show that  $r_R((p,0)R) = (0,p)R$ , so  $r_R((p,0)R)$  dose not contain a nonzero idempotent of  $R$ ; and hence  $R$  is not  $n$ -generalized right quasi-Baer, for any positive integer  $n$ .

**Lemma 1.3.** Let  $R$  be an abelian  $n$ -generalized right p.q.-Baer ring, then  $r_R(I^n) = r_R(I^m)$  for every principal right ideal  $I$  of  $R$  and each positive integer  $m$  with  $n \leq m$ .

**Proof.** It is enough to show that  $r_R(I^n) = r_R(I^{n+1})$ . Let  $x \in r_R(I^{n+1})$ , then  $Ix \subseteq r_R(I^n) = fR$  for some idempotent  $f \in R$ . Hence  $I^n x = I^n x f = 0$ . Thus  $x \in r_R(I^n)$ .

## 2. $\mathbf{N}$ -generalized right principally quasi Baer subrings of the matrix rings

In this section we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. Theorem 2.3, which extends [18, Proposition 6], enables us to provide more examples of matrix rings that are both  $n$ -generalized p.q.-Baer and  $n$ -generalized p.p.-ring. We begin with Theorem 2.2 below, which enables us to generate examples of  $n$ -generalized p.q.-Baer subrings of the matrix ring  $M_n(R)$ :

**Lemma 2.1**[18, Lemma 2]. Let  $R$  be an abelian ring and define

$$S_n := \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\},$$

with  $n$  a positive integer  $\geq 2$ . Then every idempotent in  $S_n$  is of the form

$$\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix} \text{ with } f^2 = f \in R$$

We will use  $S_n$  Throughout the remainder of the paper, to denote the ring indicated in Lemma 2.1.

**Theorem 2.2.** If  $R$  is an abelian p.q.-Baer ring and  $n (\geq 2)$  is a positive integer, then  $S_n$  is an  $n$ -generalized right p.q.-Baer ring.

**Proof.** We proceed by induction on  $n$ . It is easy to show that  $S_2$  is a 2-generalized right p.q.-Baer ring. Let  $I_n$  be a principal right ideal of  $S_n$ . Consider  $I_{n-1,1} = \{B \in S_{n-1} \mid B \text{ is obtained by deleting } n\text{-th row and } n\text{-th column of a matrix in } I_n\}$ , and  $I_{n-1,2} = \{B \in S_{n-1} \mid B \text{ is obtained by deleting } 1\text{-th row and } 1\text{-th column of a matrix in } I_n\}$ . It is clear that  $I_{n-1,1}$  and  $I_{n-1,2}$  are principal right ideals of  $S_{n-1}$ . By induction hypothesis and Lemma 2.1, there are  $e_i^2 = e_i \in S_{n-1}, f_i^2 = f_i \in R$  for  $i = 1, 2$  such that  $r_{S_{n-1}}(I_{n-1,i}^{n-1}) = e_i S_{n-1}$ ,  $e_i = \begin{pmatrix} f_i & 0 & \cdots & 0 \\ 0 & f_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_i \end{pmatrix}$ . Let  $J$  be the set of entries of the main diagonal of the elements of  $I_{n-1,1}$  or  $I_{n-1,2}$ . It is clear that  $J$  is a principal right ideal of  $R$ . Since  $R$  is right p.q.-Baer,  $r_R(J) = f_1 R = f_2 R$ . Hence  $f_1 = f_2$ , since  $R$  is an abelian ring. Now let

$$X = \begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix} \in r_{S_n}(I_n^n) \text{ and } Y = \begin{pmatrix} a_1 a_2 a_3 \cdots a_n & y_{12} & \cdots & y_{1n} \\ 0 & a_1 a_2 a_3 \cdots a_n & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 a_2 a_3 \cdots a_n \end{pmatrix} \in I_n^n$$

Since  $r_{S_{n-1}}(I_{n-1,1}^{n-1}) = r_{S_{n-1}}(I_{n-1,2}^{n-1}) = e_1 S_{n-1}$ ,  $x$  and  $x_{ij}$ 's are in  $f_1 R$  for each  $i$  and  $j$  except  $x_{1n}$ . So we have  $a_1 a_2 \cdots a_n x_{1n} + y_{1n} x = 0$ . Hence  $y_{1n} x = 0$ , since  $f_1 \in B(R)$ . Thus  $x_{1n} \in f_1 R$  and hence  $r_{S_n}(I_n^n) \subseteq e S_n$  for

$$e = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_1 \end{pmatrix} \in S_n.$$

Since, for each  $Y \in I_n e$ , all entries of the main diagonal of  $Y$  are zero and  $e$  is central,  $I_n^n e = (I_n e)^n = 0$ . Thus  $r_{S_n}(I_n^n) = e S_n$ . Therefore  $S_n$  is  $n$ -generalized right p.q.-Baer.

The following result, which generalizes [18, Proposition 6], provides examples of

matrix rings that are both  $n$ -generalized p.q.-Baer and  $n$ -generalized p.p.-ring:

**Theorem 2.3.** If  $R$  is an abelian p.p.-ring, then  $S_n$  is an abelian  $n$ -generalized p.p.-ring.

**Proof.** We prove by induction on  $n$ . First, we show that the trivial extension  $S_2$  of  $R$  is

2-generalized right p.p. Let  $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in S_2$  and  $r_R(a) = eR$ , with  $e = e^2 \in R$ . It is clear

that,  $fR \subseteq r_{S_2}(A^2)$  with  $f = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$ . Next, let  $A^2 \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = 0$ . Since  $R$  is reduced,

$a^2x = ax = 0$  and  $a^2y = ay = 0$ . Hence  $ex = x$  and  $y = ey$ . Thus  $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = f \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ .

Therefore  $S_2$  is 2-generalized right p.p. Now assume  $B = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in S_n$ .

Consider  $B_1 = \begin{pmatrix} a & a_{12} & \cdots & a_{1n-1} \\ 0 & a & \cdots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$  and  $B_2 = \begin{pmatrix} a & a_{23} & \cdots & a_{2n} \\ 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$  in  $S_{n-1}$ , then by the

induction hypothesis, there exists  $e_i^2 = e_i \in S_{n-1}$ ,  $f_i^2 = f_i \in R$ , such that  $r_{S_{n-1}}(B_i^{n-1}) = e_i S_{n-1}$ ,

$e_i = \begin{pmatrix} f_i & 0 & \cdots & 0 \\ 0 & f_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_i \end{pmatrix}$  for  $i = 1, 2$ . By direct calculations, we have  $r_{S_n}(B^{2n-2}) = e S_n$  with

$e = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$ . Since  $r_R(a) = fR$ , by [27, Lemma 3],  $r_{S_n}(B^n) = r_{S_n}(B^{2n-2}) = e S_n$ .

**Corollary 2.4** [18, Proposition 6]. If  $R$  is a domain, then  $S_n$  is an abelian  $n$ -generalized p.p.-ring.

For a semicommutative ring, the definitions of  $n$ -generalized right p.q.-Baer and  $n$ -generalized right p.p. are coincide:

**Proposition 2.5.** Let  $R$  be a semicommutative ring. Then  $R$  is  $n$ -generalized right p.q.-Baer if and only if  $R$  is  $n$ -generalized right p.p.

**Proof.** Let  $R$  be  $n$ -generalized right p.q.-Baer and  $a \in R$ . Then  $r_R(aR)^n = eR$  for some idempotent  $e \in R$ . Let  $x \in r_R(a^n)$ . Since  $R$  is semicommutative,  $RaRx \subseteq r_R(a^{n-1})$ , which implies that  $r_R(aR)^n = eR$ . The converse is similar.

There exists an  $\mathfrak{n}$ -generalized right p.q.-Baer ring, which is generalized p.p.-ring but is not semicommutative.

**Example 2.6.** Let  $R$  be an integral domain and  $S_4$  be defined over  $R$ . Then  $S_4$  is abelian 4-generalized p.p.-ring and is 4-generalized p.q.-Baer by Corollary 2.4. By considering  $b = a = e_{12} + e_{14} + e_{34}$  and  $c = e_{23}$  in  $S_4$ , where  $e_{ij}$  denote the matrix units, we have  $ab = 0$ , and  $acb \neq 0$ , hence  $aS_4b \neq 0$ .

Now we conjecture that subrings of  $\mathfrak{n}$ -generalized right p.q.-Baer rings are also  $\mathfrak{n}$ -generalized right p.q.-Baer. But the answer is negative by the following.

**Example 2.7.** For a field  $F$ , take  $F_n = F$  for  $n = 1, 2, \dots$ , and let  $S$  be the  $2 \times 2$  matrix ring over the ring  $\prod_{n=1}^{\infty} F_n$ . By [7, Proposition 2.1 and Theorem 2.2] we have that  $S$  is a p.q.-Baer ring. Let

$$R = \left( \begin{array}{cc} \prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & \langle \bigoplus_{n=1}^{\infty} F_n, 1 \rangle \end{array} \right),$$

which is a subring of  $S$ , where  $\langle \bigoplus_{n=1}^{\infty} F_n, 1 \rangle$  is the  $F$ -algebra generated by  $\bigoplus_{n=1}^{\infty} F_n$  and 1. Then by [7, Example 1.6],  $R$  is semiprime p.p which is neither right p.q.-Baer (and hence not  $\mathfrak{n}$ -generalized right p.q.-Baer), nor left p.q.-Baer (and hence not  $\mathfrak{n}$ -generalized left p.q.-Baer).

### 3. Examples of $\mathfrak{n}$ -generalized p.q.-Baer subrings

Although the class of  $\mathfrak{n}$ -generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, all quasi-Baer rings, and all abelian p.p. rings), however we show by examples that the class of  $\mathfrak{n}$ -generalized p.q.-Baer rings properly extends the aforementioned classes.

By the following example, there is an abelian p.q.-Baer (hence semiprime) ring  $R$ ,

which is not reduced, but  $S_n$  is an abelian  $n$ -generalized right p.q.-Baer ring that is not semiprime.

**Example 3.1.** By Zaleskii and Neroslavskii [10, Example 14.17, p.179], there is a simple noetherian ring  $R$  that is not a domain and in which 0 and 1 are the only idempotents. Thus  $R$  is an abelian p.q.-Baer ring that is neither left nor right p.p., and hence is not reduced. By [7, Proposition 1.17]  $R$  is semiprime and by Theorem 2.1,  $S_n$  is abelian  $n$ -generalized p.q.-Baer, that is not semiprime and hence is not right p.q.-Baer.

**Example 3.2.** If  $R$  is an abelian p.q.-Baer ring, then  $R[x]/\langle x^3 \rangle$  is an  $n$ -generalized p.q.-Baer ring.

**Proof.** First we note that  $\Theta : T \rightarrow R[x]/\langle x^3 \rangle$  defined by

$$(a_0, a_1, a_2) \rightarrow (a_0 + a_1x + a_2x^2) + \langle x^3 \rangle$$

is an isomorphism, where  $T = \{(a, b, c) \mid a, b, c \in R\}$  is a ring with addition componentwise and the multiplication defined by

$$(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = (a_1a_2, a_1b_2 + b_1a_2, a_1c_2 + b_1b_2 + c_1a_2).$$

Let  $J$  be an ideal of  $T$ . Suppose  $I = \{a \in R \mid (a, b, c) \in J\}$ , it is clear that  $I$  is an ideal of  $R$ . Since  $R$  is p.q.-Baer,  $r_R(I) = eR$  for an idempotent  $e \in R$ . We can show that  $r(J^3) = (e, 0, 0)T$ , and hence, the result follows.

There exists a commutative  $n$ -generalized p.q.-Baer (hence  $n$ -generalized p.p.-) ring  $R$ , over which  $S_n$  is not an  $n$ -generalized p.p.-ring.

**Example 3.3.** Let  $p \neq 3$  be a prime integer and  $Z_{p^3}$  be the ring of integers modulo  $p^3$ , and  $S_3$  be defined over  $Z_{p^3}$ . Let  $A = pI_3 + e_{13}$ , where  $I_3$  is the identity matrix and  $e_{ij}$  denote the matrix units. It is clear that  $pI_3 + e_{13} + e_{12} \in r_{S_n}(A^3)$  and idempotents of  $S_3$  are  $I_3$  and 0. Hence  $r_{S_3}(A^3) \neq I_3S_3$  and that  $S_3$  is not 3-generalized p.p.-ring, but  $Z_{p^3}$  is a 3-generalized p.p.-ring.

**Example 3.4.** For every abelian quasi-Baer (resp. p.p.-) ring  $R$ , by Theorems 2.1 and



2.2, the ring  $S_n$  is  $n$ -generalized right p.q.-Baer, which is not right p.q.-Baer. Therefore we are able to provide examples of  $n$ -generalized right p.q.-Baer rings that is not right p.q.-Baer:

Let  $F$  be a field, and  $R = F[x]$  be the polynomial ring where  $x$  is an indeterminate. Then  $S_n$  is a  $n$ -generalized right p.q.-Baer ring that is not right p.q.-Baer.

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