Contractibility and idempotents in Banach algebras

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Abstract

Let A be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of b-contractibility and one of its equivalent forms are introduced. Through an example, it is shown that b-contractibility is strictly weaker than contractibility.

Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4],[5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call b-contractibility. We give a characterization of b-contractibility analog to that of contractibility given by Taylor. Also, we show that b-contractibility is strictly weaker than contractibility.

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First we recall some terminology. Throughout this paper, A is a Banach algebra and A-module means Banach A-bimodule. For a subset E of A, E' is the commutant of E. If for every A-bimodule X every bounded derivation from A into X is inner, then A is called *contractible*. Also, the term "semisimple" means $Jacobson\ semisimple$. An idempotent $e \in A$ is called minimial if eAe is a division ring. If e and e are idempotents in e0, we write e1 if e1 fe = ef1 e holds. A nonzero idempotent e1 is called e2 implies that e3 or e4. Also, two idempotents e5 and e5 are said to be e5 or e6 in e7. The e8 right e9 in e9 in e9. Let e9 be a subset of e9. The e9 right e9 in e9 in e9 in e9 is the set

$$ran(S) = \{a \in A : ba = 0 \text{ for } b \in S\}.$$

The left annihilator lan(S) is defined semilarly. The *annihilator* of S is the set $Ann(S) = ran(S) \cap lan(S)$.

Contractibility

Theorem 2.1. Let A be a contractible Banach algebra which is an ideal in a Banach algebra B. Then A + A' = B.

Proof. If $A + A' \neq B$, then we can choose $b \in B - (A + A')$. Now define

$$D: A \to A, x \mapsto xb - bx$$
.

Clearly D is a derivation on A. By assumption there exists an $a \in A$ such that D(x) = xa - ax for all $x \in A$. The latter result implies that $b - a \in A$ or equivalently $b \in A + A$ which contradicts the selection of b. Therefore A + A' = B.

Theorem 2.2. Let A be a contractible Banach algebra which is an ideal in a Banach algebra B. Then $B = A \oplus \text{Ann}(A)$.

Proof. Since A is contractible then $M_2(A)$ with l^1 -norm is contarctible, where $M_2(A)$ is the algebra of 2×2 matrices with the enteries from A. On the other hand $M_2(A)$ is an ideal in $M_2(B)$ and by Theorem 2.1 we have the equality $M_2(B) = M_2(A) + M_2(A)$. One can easily observe that

$$M_2(A)' = \begin{bmatrix} A' & Ann(A) \\ Ann(A) & A' \end{bmatrix}.$$

Thus B = A + Ann(A). But $A \cap Ann(A) = 0$, because A is unital. Therefore the identity $B = A \oplus Ann(A)$ holds.

Remark. In Theorems 2.1 and 2.2, A and B are related only algebrically. Indeed if there exists an infinite dimensional contractible Banach algebra A which is an ideal in a Banach algebra B, then the norm topology of A could be different from the relative norm topology of A which inherits from B.

Theorem 2.3. Let A be a contractible Banach algebra which admits a nonzero multiplicative linear functional f. Then A contains a central minimal idempotent.

Proof. Let $d = \sum_{n=1}^{\infty} a_n \otimes b_n$ be a diagonal for A and define

$$T: A, \to a \mapsto \sum_{n=1}^{\infty} \langle f, aa_n \rangle b_n.$$

Since $\sum_{n} a_{n}b_{n} = 1$, then $< f, T(1) > = < f, \sum_{n} < f, a_{n} > b_{n} > = \sum_{n} < f, a_{n} > < f, b_{n} >$ $= \sum_{n} < f, a_{n}b_{n} > = < f, \sum_{n} a_{n}b_{n} > = < f, 1 > = 1.$

Thus $T(1) \neq 0$. Moreover for every $a \in A$ and $g, h \in A^*$ we have

$$\langle h, \sum_{n} \langle g, aa_{n} \rangle b_{n} \rangle = \sum_{n} \langle g, aa_{n} \rangle \langle h, b_{n} \rangle$$

$$= \langle g \otimes h, \sum_{n} aa_{n} \otimes b_{n} \rangle$$

$$= \langle g \otimes h, \sum_{n} a_{n} \otimes b_{n} a \rangle$$

$$= \sum_{n} \langle g, a_{n} \rangle \langle h, b_{n} a \rangle$$

$$= \langle h, \sum_{n} \langle g, a_{n} \rangle b_{n} a \rangle.$$

This implies that

$$\sum_{n} < g, aa_{n} > b_{n} = \sum_{n} < g, a_{n} > b_{n}a.$$

Thus we assume that

T(1)=e, then we have $T(a) = \sum_n \langle f, aa_n \rangle b_n = \sum_n \langle f, a_n \rangle b_n a = ea$. On the other hand we have $T(a) = \sum_n \langle f, aa_n \rangle b_n = \langle f, a \rangle \sum_n \langle f, a_n \rangle b_n = \langle f, a \rangle e$. Hence T is an operator of rank one and $e^2 = T(e) = \langle f, e \rangle e = e$. Now define

$$T_1: A \to A, a \mapsto \sum_n a_n < f, aa_n > .$$

With a similar argument we can show that

$$T_1(a) = ae' = \langle f, a \rangle e' \quad a \in A$$

where $e' = T_1(1)$. Also we have $e'^2 = e'$ and $\langle f, e' \rangle = 1$. Now the identities

$$ee' = < f, e' > e = e, \qquad ee' = < f, e > e' = e'$$

imply that e = e' and for every $a \in A$ we have

$$ea = \langle f, a \rangle e = \langle f, a \rangle e' = ae' = ae.$$

Therefore e is a central idempotent. In addition since T is a rank one operator and ranT = eAe, then eA = eAe = Ce is a division ring. Therefore e is a minimal idempotent.

b-Contractibility

Definition. Let A be a Banach algebra and π be the natural map,

$$\pi: A \otimes A \longrightarrow A, \quad \pi(\sum_n a_n \otimes b_n) \rightarrow \sum_n a_n b_n.$$

Let $b \in A$ and X be an A-module. We say that a derivation $\delta A \longrightarrow X$ is a b-derivation if there exists another derivation $\delta' A \longrightarrow X$ such that $\delta = b\delta'$, where $(b\delta')(a) = b\delta'(a)$. Also we say that A is b-contractible if for every A-module X, every bounded b-derivation from A into X is inner. We call $d \in A \hat{\otimes} A$ a b-diagonal if $\pi(d) = b$ and a.d = d.a for all $a \in A$.

Theorem 3.1. Let A be a unital Banach algebra and $b \in A' - \{0\}$. Then A is b-contractible if and only if A has a b-diagonal.

Proof. First suppose A is b-contractible and π is defined as above. Clearly $\ker \pi$ is an A-module and if we define

$$\delta: A \to \ker \pi, a \mapsto ab \otimes 1 - b \otimes a$$

then it is easy to see that δ is a *b*-derivation. Indeed $\delta = b\delta$ where

$$\delta': A \to \ker \pi, a \mapsto a \otimes 1 - 1 \otimes a$$

ince A is b-contractible, then threre exists an element $\sum_n c_n \otimes d_n \in \ker \pi$ such that

$$\delta(a) = \sum_{n} ac_{n} \otimes d_{n} - \sum_{n} c_{n} \otimes d_{n} a \quad a \in A.$$

Let $d = b \otimes 1 - \sum_n c_n \otimes d_n$. The above identities show that $\pi(d) = b$ and a.d = d.a for all $a \in A$. Therefore, d is a b-diagonal for A. Conversely suppose $d = \sum_n a_n \otimes b_n$ is a b-diagonal for A, X is an A-module and $\delta: A \longrightarrow X$ is a bounded derivation. Clearly the map

$$\psi: A \times A \rightarrow X, (a,c) \mapsto a\delta(c)$$

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map $\varphi: A \hat{\otimes} A \longrightarrow X$ such that $\varphi \circ \otimes = \psi$ that is $\varphi(a \otimes c) = a\delta(c)$. In particular using the fact that d is a b-diagonal for A, we get

$$\sum_{n} a a_{n} \delta(b_{n}) = \varphi(a.d) = \varphi(d.a) = \sum_{n} a_{n} \delta(b_{n}a), \quad a \in A.$$

Now if $x = \sum_{n} a_n \delta(b_n)$, then for every $a \in A$ we have

$$ax - xa = \sum_{n} aa_{n}\delta(b_{n}) - \sum_{n} a_{n}\delta(b_{n})a = \sum_{n} aa_{n}\delta(b_{n}) + b\delta(a) - \sum_{n} a_{n}\delta(b_{n}a).$$

Thus the identity $ax - xa = b\delta(a)$ holds for every $a \in A$. Therefore every b-derivation is inner.

Example 3.2. Let A be the Banach algebra $l_1(N)$ with pointwise multiplication and $\{e_n\}$ be the standard basis for A. Then for every positive integer n, A is e_n -contractible. Indeed $e_n \otimes e_n$ is an e_n -diagonal for A. But A is not contractible, since it is not unital. Therefore b-contractibility dose not imply contractibility.

Remark. If A is contractible, then it is unital and one can easily observe that A is b-contractible for every $b \in A - \{0\}$. However the above example shows that for non-unital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

Problem. Does there exist a unital Banach algebra which is b-contactible for some nonzero central idempotent b, but is not contractible?

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