

## On pointwise inner automorphisms of nilpotent groups of class 2

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### Abstract

An automorphism  $\theta$  of a group  $G$  is pointwise inner if  $\theta(x)$  is conjugate to  $x$  for any  $x \in G$ . The set of all pointwise inner automorphisms of group  $G$ , denoted by  $\text{Aut}_{\text{pwi}}(G)$  form a subgroups of  $\text{Aut}(G)$  containing  $\text{Inn}(G)$ . In this paper, we find a necessary and sufficient condition in certain finitely generated nilpotent groups of class 2 for which  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ . We also prove that in a nilpotent group of class 2 with cyclic commutator subgroup  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$  and the quotient  $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$  is torsion. In particular if  $G'$  is a finite cyclic group then  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ .

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### Introduction

By definition, a pointwise inner automorphism of a group  $G$  is an automorphism  $\theta: G \rightarrow G$  such that  $t$  and  $\theta(t)$  are conjugate for any  $t \in G$ . This notion appears in the famous book of Burnside [1, Note B, p 463]. Denote by  $\text{Aut}_{\text{pwi}}(G)$  the set of all pointwise inner automorphisms of  $G$ .

Obviously,  $\text{Aut}_{\text{pwi}}(G)$  contains  $\text{Inn}(G)$ , the group of all inner automorphisms of  $G$ . These groups can coincide, for instance when  $G$  is  $S_n, A_n, \text{SL}_n(D)$  and  $\text{GL}_n(D)$  where  $D$  is an Euclidean domain (see [7], [10], [11]).

By a result of Grossman [5], it turns out that  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$  when  $G$  is a free group. Endimioni in [4] proved that this property remains true in a free nilpotent group.

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Also Yadav in [12] gave a sufficient condition for a finite  $p$ -group  $G$  of nilpotent class 2 to be such that  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ . But the equality does not hold in general.

In fact, in 1911, Burnside posed the following question: Does there exist any finite group  $G$  such that  $G$  has a non-inner and pointwise inner automorphism? In 1913, Burnside himself gave an affirmative answer to this question [3]. Indeed, there are many examples of groups admitting a pointwise inner automorphism which is not inner (see, for instance [3], [4], [8], [9], [12] where these groups are besides nilpotent).

Segal also gave a subtle example. He constructed a finitely generated torsion-free nilpotent group  $G$ , in which  $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$  contains an element of infinite order (see [9]).

In this paper we study the pointwise inner automorphisms of a finitely generated nilpotent group of class 2 with cyclic commutator subgroup.

We introduce the following definition:

**Definition.** Let  $G$  be a finitely generated nilpotent group of class 2. Then  $G/Z(G)$  is finitely generated abelian group and thus  $G/Z(G) = \langle x_1Z(G) \rangle \times \dots \times \langle x_kZ(G) \rangle$  for some  $x_1, \dots, x_k \in G$ . The group  $G$  is called **d**-group if the following distributive law holds in  $G$ ,

$$[x_1^{\alpha_1} \dots x_k^{\alpha_k}, G] = [x_1, G]^{\alpha_1} \dots [x_k, G]^{\alpha_k}$$

where  $\alpha_i \in \mathbb{Z}$  and  $1 \leq i \leq k$ .

Let  $G$  be a 2-generator nilpotent group of class 2. It is straightforward to show that  $G$  is a **d**-group.

To give an example of an infinite **d**-group, consider the group  $G$  with the following presentation

$$G = \langle x_1, x_2, x_3, x_4, x: [x_i, x_j] = x^{m_{ij}}, [x_i, x] = 1; 1 \leq i \leq 4 \text{ and } i < j \rangle,$$

where  $m_{i,i+1} = 1$  for all  $1 \leq i < 4$  and  $m_{ij} = 0$  for all  $i + 1 < j$ . Then  $G' = Z(G) = \langle x \rangle \simeq \mathbb{Z}$  and  $G/Z(G) = \langle \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \rangle \simeq \mathbb{Z}^4$ . A quick calculation shows that

$$[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, G] = [x_1, G]^{\alpha_1} [x_2, G]^{\alpha_2} [x_3, G]^{\alpha_3} [x_4, G]^{\alpha_4} = \langle x^\alpha \rangle,$$

Where  $\alpha_i \in \mathbb{Z}$  for all  $1 \leq i \leq 4$  and  $\alpha = \text{gcd}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ . Therefore  $G$  is an infinite **d**-

group.

Now we give a nilpotent group  $G$  of class 2 which is not a  $d$ -group.

Let  $G$  be a free nilpotent group of class 2 on 4 generators  $a_1, a_2, a_3$  and  $a_4$ . If  $c_{ij} = [a_i, a_j]$  for  $1 \leq i < j \leq 4$ , then the relations in  $G$  are  $[c_{ij}, a_k] = 1$  for  $1 \leq i < j \leq 4$  and  $1 \leq k \leq 4$ , and their consequences. Macdonald in [6] proved that  $c_{13}c_{24}$  is not a commutator. Therefore  $G$  is not a  $d$ -group.

**Theorem 1.** Let  $G$  be a finitely generated nilpotent group of class 2 and

$$G/Z(G) = \langle \bar{x}_1 \rangle \times \dots \times \langle \bar{x}_k \rangle.$$

- (i) There exists a monomorphism  $\text{Aut}_{\text{pwi}}(G) \hookrightarrow \prod_{i=1}^k \text{Hom}(\langle \bar{x}_i \rangle, [x_i, G])$ .
- (ii) If  $[x_i, G]$  is cyclic for all  $1 \leq i \leq k$ , then there exists a monomorphism  $\text{Aut}_{\text{pwi}}(G) \hookrightarrow \text{Inn}(G)$ .

In particular if  $G$  is a  $d$ -group of class 2 then the monomorphisms in (i) and (ii) are isomorphism. Furthermore  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$  if and only if  $[x_i, G]$  is cyclic for all  $1 \leq i \leq k$ .

Notice that if  $G$  is a finite group then, as consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

In particular, we derive the following consequences of Theorem 1.

**Corollary 1.** Let  $G$  be a finitely generated nilpotent group of class 2 in which  $G'$  is cyclic, then  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ . In particular if  $G'$  is finite, then  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ .

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if  $G'$  is cyclic then  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ . But we cannot hope for a similar conclusion when  $G$  is not finite. We will provide an example in the section 2. However, in a finitely generated nilpotent group of class 2, by Corollary 1 we have  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ . So the structure of  $\text{Aut}_{\text{pwi}}(G)$  is determined.

**Corollary 2.** Let  $G$  be a finitely generated nilpotent group of class 2. If the commutator subgroup of  $G$  is cyclic, then  $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$  is torsion.

Let  $G$  be a group and  $N$  be a non-trivial proper normal subgroup of  $G$ . The pair

$(G, N)$  is called a Camina pair if  $xN \subseteq x^G$  for all  $x \in G \setminus N$ . A group  $G$  is called a Camina group if  $(G, G')$  is a Camina pair.

Clearly, if  $G$  is a Camina group of class 2 then it is a  $d$ -group. So, as an immediate consequence of Theorem 1, one readily gets the following corollary.

**Corollary 3.** Let  $G$  be a finitely generated nilpotent group of class 2. If  $G$  is a Camina group then  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$  if and only if  $G'$  is cyclic. Particularly, if  $G/Z(G)$  is finite, then  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$  if and only if  $G'$  is cyclic.

### Preliminary results

Our notation is standard. Let  $G$  be a group, by  $C_m, G'$  and  $Z(G)$ , we denote the cyclic group of order  $m$ , the commutator subgroup and the center of  $G$ , respectively.

If  $x, y \in G$ , then  $x^y$  denotes the conjugate element  $y^{-1}xy \in G$ . For  $x \in G$ ,  $x^G$  denotes the conjugacy class of  $x$  in  $G$ . The commutator of two elements  $x, y \in G$  is defined by  $[x, y] = x^{-1}y^{-1}xy$  and more generally, the left-normed commutator of  $n$  elements  $x_1, \dots, x_n$  is defined inductively by

$$[x_1, \dots, x_{n-1}, x_n] = [x_1, \dots, x_{n-1}]^{-1}x_n^{-1}[x_1, \dots, x_{n-1}]x_n.$$

If  $H \leq G$ ,  $[x, H]$  denotes the set of all  $[x, h]$  for  $h \in H$ , this is a subgroup of  $G$  when  $G$  is of class 2. For any group  $H$  and abelian group  $K$ ,  $\text{Hom}(H, K)$  denotes the group of all homomorphisms from  $H$  to  $K$ . Also  $C^*$  is the set of all central automorphisms of  $G$  fixing  $Z(G)$  elementwise.

Yadav in [12] shows that in a finite nilpotent group of class 2, there exists a monomorphism from  $\text{Aut}_{\text{pwi}}(G)$  into  $\text{Hom}(G/Z(G), G')$ . It turns out that this result remains true when  $G$  is an infinite nilpotent group of class 2.

For that, let  $G$  be a nilpotent group (finite or infinite) of class 2. Let  $\alpha \in \text{Aut}_{\text{pwi}}(G)$ . Then the map  $g \mapsto g^{-1}\alpha(g)$  is a homomorphism from  $G$  into  $G'$ . This homomorphism sends  $Z(G)$  to 1. So it induces a homomorphism  $f_\alpha: G/Z(G) \rightarrow G'$ , sending  $\bar{g} = gZ(G)$  to  $g^{-1}\alpha(g)$ , for any  $g \in G$ . Define

$$\text{Hom}_{\text{pwi}}(G/Z(G), G') = \{f \in \text{Hom}\left(\frac{G}{Z(G)}, G'\right) : f(\bar{g}) \in [g, G] \text{ for all } g \in G\}.$$

To prove  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Hom}_{\text{pwi}}(G/Z(G), G')$ , we use the following well-known result.

**Lemma 1.1** Let  $N$  be a normal subgroup of a group  $G$ . Let  $\theta$  be an endomorphism of  $G$  such that  $\theta(N) \leq N$ . Denote by  $\bar{\theta}$  and  $\theta_0$  the endomorphisms induced by  $\theta$  in  $G/N$  and  $N$ , respectively. If  $\bar{\theta}$  and  $\theta_0$  are surjective (injective), then so is  $\theta$ .

**Proposition 1.2** Let  $G$  be a nilpotent group of class 2. Then the above map  $\varphi: \alpha \mapsto f_\alpha$  is an isomorphism from  $\text{Aut}_{\text{pwi}}(G)$  into  $\text{Hom}_{\text{pwi}}(G/Z(G), G')$ .

Proof. Since for any  $\alpha \in \text{Aut}_{\text{pwi}}(G)$ , by the definition  $f_\alpha \in \text{Hom}_{\text{pwi}}(G/Z(G), G')$ ,  $\varphi$  is well defined. Let  $\alpha_1, \alpha_2 \in \text{Aut}_{\text{pwi}}(G)$  and  $g \in G$ . We have  $\alpha_1(g^{-1}\alpha_2(g)) = g^{-1}\alpha_2(g)$ , since  $g^{-1}\alpha_2(g) \in G' \leq Z(G)$ . This implies that

$$\begin{aligned} f_{\alpha_1\alpha_2}(\bar{g}) &= g^{-1}\alpha_1(\alpha_2(g)) = g^{-1}\alpha_1(gg^{-1}\alpha_2(g)) \\ &= g^{-1}\alpha_1(g) \cdot g^{-1}\alpha_2(g) = f_{\alpha_1}(\bar{g}) \cdot f_{\alpha_2}(\bar{g}). \end{aligned}$$

Hence  $\varphi$  is a homomorphism. Clearly,  $\varphi$  is injective. Now it suffices to show that  $\varphi$  is surjective.

Let  $f$  be any element of  $\text{Hom}_{\text{pwi}}(G/Z(G), G')$ . By Lemma 1.1 a quick calculation shows that  $\varphi(\alpha) = f$ , where  $\alpha$  is an element of  $\text{Aut}_{\text{pwi}}(G)$ , sending  $g \in G$  to  $gf(gZ(G))$ . Then we have  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Hom}_{\text{pwi}}(G/Z(G), G')$ .

\* Note that if  $G$  is a nilpotent group of class 2 then  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Hom}_{\text{pwi}}(G/Z(G), G')$ .

It is easy to see that in a Camina nilpotent group of class 2,  $\text{Hom}_{\text{pwi}}(G/Z(G), G') = \text{Hom}(G/Z(G), G')$ . Hence if  $G$  is a Camina group of class 2, then  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Hom}(G/Z(G), G')$ .

The following well-known facts will be used repeatedly.

**Lemma 1.3** Let  $A, B$  and  $C$  be abelian groups.

- (i)  $\text{Hom}(A \times B, C) \simeq \text{Hom}(A, C) \times \text{Hom}(B, C)$ .
- (ii)  $\text{Hom}(A, B \times C) \simeq \text{Hom}(A, B) \times \text{Hom}(A, C)$ .
- (iii)  $\text{Hom}(C_m, C_n) \simeq C_d$  where  $d = \text{gcd}(m, n)$ .
- (iv)  $\text{Hom}(\mathbb{Z}, A) \simeq A$ .
- (v) If  $A$  is torsion group and  $B$  is torsion-free group, then  $\text{Hom}(A, B) = 1$ .
- (vi) If  $\text{gcd}(|A|, |B|) \neq 1$ , then  $\text{Hom}(A, B) \neq 1$ .

## Main Result

Let  $G$  be a finite abelian group. We denote by  $G_p$ , the  $p$ -primary component of  $G$ . Hence  $G = \prod_{p \in \pi(G)} G_p$  where  $\pi(G)$  denotes the set of all primes  $p$  dividing  $|G|$ .

To prove Theorem 1, we need the following Lemma.

**Lemma 2.1** ([1, Corollary 1.4]) Let  $A$  and  $B$  be two finite abelian groups and  $\exp(A) | \exp(B)$ . Then  $\text{Hom}(A, B) \simeq A$  if and only if  $B \simeq C_m \times H$  in which  $C_m \simeq \prod_{p \in \pi(A)} B_p$  and  $H \simeq \prod_{p \notin \pi(A)} B_p$ . In particular, if  $\pi(A) = \pi(B)$  then this is equivalent to  $B$  is a cyclic group.

Let  $G$  be a finitely generated nilpotent group of class 2. Then  $G/Z(G)$  is finitely generated abelian group and thus  $G/Z(G) = \langle x_1 Z(G) \rangle \times \dots \times \langle x_k Z(G) \rangle$  for some  $x_1, \dots, x_k \in G$ .

Let  $f \in \text{Hom}_{\text{pwi}}(G/Z(G), G')$ . So  $f(gZ(G)) \in [g, G]$  for all  $g \in G$ . In particular, for all  $1 \leq i \leq k$  we have  $f(x_i Z(G)) \in [x_i, G]$ . Now we prove Theorem 1.

### Proof of Theorem 1.

(i) By Proposition 1.2, we have  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Hom}_{\text{pwi}}(G/Z(G), G')$ . It suffices to show that there exists a monomorphism from  $\text{Hom}_{\text{pwi}}(G/Z(G), G')$  into  $\prod_{i=1}^k \text{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$ . Let  $f \in \text{Hom}_{\text{pwi}}(G/Z(G), G')$ . Denote by  $f_i$ , the homomorphism induced by  $f$  in  $\langle \overline{x_i} \rangle$ , for all  $1 \leq i \leq k$ . Since  $G$  is a nilpotent group of class 2, we have  $[a^m, b] = [a, b]^m = [a, b^m]$  for each  $a, b \in G$  and  $m \in \mathbb{Z}$ . Consequently,  $[x_i^m, G] \leq [x_i, G]$  for all  $m \in \mathbb{Z}$  and  $1 \leq i \leq k$ . Therefore  $f_i \in \text{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$ . Thus the map  $\alpha$  sending any  $f \in \text{Hom}_{\text{pwi}}(G/Z(G), G')$  to  $\alpha(f) = (f_1, \dots, f_k) \in \prod_{i=1}^k \text{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$  is well defined. Now we prove that this map is a monomorphism. Since  $(fg)_i = f_i g_i$  for each  $f, g \in \text{Hom}_{\text{pwi}}(G/Z(G), G')$  and  $1 \leq i \leq k$ ,  $\alpha$  is homomorphism. Clearly,  $\ker \alpha$  is trivial, this implies that  $\alpha$  is monomorphism. Hence the proof of (i) is complete.

(ii) First we show that  $[x_i, G]$  is finite if and only if  $\langle \overline{x_i} \rangle$  is finite, and further

$\exp([x_i, G]) = \exp(\langle \bar{x}_i \rangle) = |\bar{x}_i|$ . For this, let  $|[x_i, G]| = n$ . Since  $G$  is a nilpotent group of class 2, we have  $[x_i^n, g] = [x_i, g]^n = 1$  for all  $g \in G$  and so  $x_i^n \in Z(G)$ . Hence  $\langle \bar{x}_i \rangle$  is finite and  $|\bar{x}_i| |n$ . Conversely if  $|\bar{x}_i| = m$  then  $x_i^m \in Z(G)$  and  $[x_i, G]^m = [x_i^m, G] = 1$ . Consequently  $[x_i, G]$  is finite and  $\exp([x_i, G]) = n|m$ . Therefore in this case,  $m = n$ . Hence by Lemma 2.1, for all  $1 \leq i \leq k$  we have  $\text{Hom}(\langle \bar{x}_i \rangle, [x_i, G]) \simeq \langle \bar{x}_i \rangle$  if and only if  $[x_i, G]$  is cyclic.

Now from (i), we have a monomorphism from  $\text{Aut}_{\text{pwi}}(G)$  into  $\prod_{i=1}^k \text{Hom}(\langle \bar{x}_i \rangle, [x_i, G])$  and therefore we conclude that there exists a monomorphism  $\text{Aut}_{\text{pwi}}(G) \hookrightarrow G/Z(G)$ , this completes the proof of (ii).

If  $G$  is a  $d$ -group, then it is easy to see that the monomorphism defined in (i) is an isomorphism from  $\text{Aut}_{\text{pwi}}(G)$  into  $\prod_{i=1}^k \text{Hom}(\langle \bar{x}_i \rangle, [x_i, G])$ .

Finally to complete the proof, it is sufficient to show that if  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ , then  $[x_i, G]$  is cyclic for all  $1 \leq i \leq k$ . Since  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ , by Proposition 1.2 we have  $G/Z(G) \simeq \text{Hom}_{\text{pwi}}(G/Z(G), G')$ . On the other hand,  $G$  is a  $d$ -group and hence

$$\text{Hom}_{\text{pwi}}(G/Z(G), G') \simeq \prod_{i=1}^k \text{Hom}(\langle \bar{x}_i \rangle, [x_i, G]).$$

It follows that

$$G/Z(G) = \langle \bar{x}_1 \rangle \times \dots \times \langle \bar{x}_k \rangle \simeq \prod_{i=1}^k \text{Hom}(\langle \bar{x}_i \rangle, [x_i, G]).$$

Now we may assume that  $\langle \bar{x}_1 \rangle \times \dots \times \langle \bar{x}_n \rangle$  is the torsion part and  $\langle \bar{x}_{n+1} \rangle \times \dots \times \langle \bar{x}_k \rangle$  is the torsion-free part of  $G/Z(G)$ . Since for all  $1 \leq i \leq n$ ,  $\exp([x_i, G]) = \exp(\bar{x}_i) = |\bar{x}_i|$  and  $\prod_{i=1}^n \text{Hom}(\langle \bar{x}_i \rangle, [x_i, G]) \simeq \langle \bar{x}_1 \rangle \times \dots \times \langle \bar{x}_n \rangle$ ,  $\text{Hom}(\langle \bar{x}_i \rangle, [x_i, G]) \simeq \langle \bar{x}_i \rangle$  for all  $1 \leq i \leq n$  and hence  $[x_i, G]$  is cyclic. Furthermore, we have

$$\prod_{i=n+1}^k \text{Hom}(\langle \bar{x}_i \rangle, [x_i, G]) \simeq \langle \bar{x}_{n+1} \rangle \times \dots \times \langle \bar{x}_k \rangle \simeq \mathbb{Z}^{k-n}.$$

Now we have  $\text{Hom}(\langle \bar{x}_i \rangle, [x_i, G]) \simeq [x_i, G]$ , since  $\langle \bar{x}_i \rangle \simeq \mathbb{Z}$  and hence  $\prod_{i=n+1}^k [x_i, G] \simeq \mathbb{Z}^{k-n}$ . That is  $[x_i, G] \simeq \mathbb{Z}$  for all  $n + 1 \leq i \leq k$ . This implies that  $[x_i, G]$  is cyclic for all  $1 \leq i \leq k$ , as required.

\*Notice that if  $G$  is a finite group then, as a consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

The following corollary is an easy consequence of the above theorem.

**Corollary 2.2** Let  $G$  be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Then there exists a monomorphism from  $\text{Aut}_{\text{pwi}}(G)$  into  $\text{Inn}(G)$  or equivalently  $\text{Aut}_{\text{pwi}}(G)$  is isomorphic to a subgroup of  $G/Z(G)$ .

**Remark 2.3** We keep here the notation used in Theorem 1.

- (i) By the discussion of (ii) in Theorem 1, if  $G'$  is finite cyclic, then  $G/Z(G)$  is finite and  $|\text{Aut}_{\text{pwi}}(G)| \leq |\text{Inn}(G)| = |G/Z(G)|$ . On the other hand,  $\text{Inn}(G) \leq \text{Aut}_{\text{pwi}}(G)$  conclude that  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ . Note that in this case,  $G$  is not necessarily finite.
- (ii) If  $G'$  is infinite cyclic, it follows from the discussion of (ii) in Theorem 1, that  $G/Z(G)$  is a free abelian group of finite rank, say  $r(G/Z(G)) = k$ . We certainly have  $\text{Inn}(G) \leq \text{Aut}_{\text{pwi}}(G)$  and thus  $r(\text{Inn}(G)) \leq r(\text{Aut}_{\text{pwi}}(G))$ . Also  $r(\text{Aut}_{\text{pwi}}(G)) \leq r(\text{Inn}(G))$ , since  $\text{Aut}_{\text{pwi}}(G)$  is isomorphic to a subgroup of  $\text{Inn}(G)$ . Therefore  $\text{Aut}_{\text{pwi}}(G)$  and  $\text{Inn}(G)$  have the same rank and hence  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ .

Now it is easy to deduce Corollary 1 from Remark 2.3.

**Remark 2.4** It is known that in a nilpotent groups of class 2,  $\text{Inn}(G) \leq \text{Aut}_{\text{pwi}}(G) \leq C^*$ . So  $\text{Inn}(G) = \text{Aut}_{\text{pwi}}(G)$  when  $\text{Inn}(G) = C^*$ . In [1] we characterized all non torsion-free finitely generated groups in which  $\text{Inn}(G) = C^*$ . We proved that  $\text{Inn}(G) = C^*$  if and only if  $G$  is an abelian group or nilpotent of class 2 and  $Z(G) \simeq C_m \times H \times \square^r$  in which  $C_m \simeq \prod_{p \in \pi(G/Z(G))} Z(G)_p$ ,  $H \simeq \prod_{p \notin \pi(G/Z(G))} Z(G)_p$  and  $r \geq 0$  is the torsion-free rank of  $Z(G)$  and  $G/Z(G)$  has finite exponent.

Hence if  $G$  is nilpotent group of class 2,  $Z(G) \simeq C_m \times H \times \square^r$  and  $G/Z(G)$  has finite exponent then we have  $\text{Inn}(G) = \text{Aut}_{\text{pwi}}(G)$ . Notice that in this case,  $G'$  is cyclic and the equality  $\text{Inn}(G) = \text{Aut}_{\text{pwi}}(G)$  also follows from Corollary 1.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if  $G'$  is cyclic then  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ . But we cannot hope for a similar conclusion when  $G$  is not finite.



For example, consider countably infinitely many copies  $H_1, H_2, \dots$  of a given nilpotent group  $H$  of class 2 with cyclic commutator subgroup. Let  $G$  (respectively,  $\bar{G}$ ) be the direct product (the cartesian product) of the family  $(H_i)_{i>0}$ . Clearly,  $G$  and  $\bar{G}$  are nilpotent of class 2. For each integer  $i > 0$ , choose an element  $a_i \in H_i$  which is not in the center of  $H_i$ . Then the inner automorphism of  $\bar{G}$  defined by  $\bar{\alpha}((t_i)_{i>0}) = (a_i^{-1}t_i a_i)_{i>0}$  induces in  $G$  a pointwise inner automorphism  $\alpha$  which is not inner (see [4]).

However, in a finitely generated nilpotent group of class 2 with cyclic commutator subgroup, we have  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ , by Corollary 1. So the structure of  $\text{Aut}_{\text{pwi}}(G)$  is determined.

Furthermore it is fairly easy to deduce Corollary 2 from Remark 2.3.

We end this part of the paper with some examples of infinite groups  $G$  satisfying the conditions of Corollary 1 and therefore  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ .

**Example 2.5** Let  $G = \langle x_1, x_2, y_1, y_2 : x_1^p = x_2^p = y_1^p = 1, [x_1, x_2] = y_1, [y_1, y_2] = [x_i, y_j] = 1; 1 \leq i, j \leq 2 \rangle$ . Then  $G$  satisfies the condition of Corollary 1. We have  $G' = \langle y_1 \rangle \simeq C_p$ ,  $Z(G) = \langle y_1, y_2 \rangle \simeq C_p \times \mathbb{Z}$  and  $G/Z(G) = \langle \bar{x}_1, \bar{x}_2 \rangle \simeq C_p \times C_p$  and hence  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ .

**Example 2.6** Let  $G = \langle x_1, x_2, x : [x_1, x_2] = x, [x_i, x] = 1; 1 \leq i \leq 2 \rangle$ . Then  $G$  satisfies the condition of Corollary 1. We have  $G' = Z(G) = \langle x \rangle \simeq \mathbb{Z}$  and  $\frac{G}{Z(G)} = \langle \bar{x}_1, \bar{x}_2 \rangle \simeq \mathbb{Z} \times \mathbb{Z}$ . Hence  $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$ . It is easy to see that in this case every pointwise inner automorphism is inner and so  $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$  (see [1, Example 3.4]).

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