Numerical Solution of Fuzzy Differential Equation by Runge-Kutta Method

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Abstract

In this paper, the numerical algorithms for solving ‘fuzzy ordinary differential equations’ are considered. A scheme based on the 4th order Runge-Kutta method is discussed in detail and it is followed by a complete error analysis. The algorithm is illustrated by solving some linear and nonlinear fuzzy Cauchy problems.

1. Introduction

The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [1]. It was followed up by D.Dubois, H. Prade in [2], who defined and used the extension principle. The fuzzy differential equation and the initial value problem were regularly treated by O. Kaleva in [5] and [6], by S. Seikkala in [7]. The numerical method for solving fuzzy differential equations is introduced by M. Ma, M. Friedman, A. Kandel in [10] by the standard Euler method. The structure of this paper organizes as follows:

In section 2 some basic results on fuzzy numbers and definition of a fuzzy derivative, which have been discussed by S. Seikkala in [10], are given. In section 3 we define the problem, this is a fuzzy Cauchy problem whose numerical solution is the main interest of this work. The numerically solving fuzzy differential equation by 4th order Runge-Kutta method is discussed in section 4. The proposed algorithm is illustrated by solving some examples in section 5 and conclusion is in section 6.

Keywords: Fuzzy Differential Equation, 4th Order Runge-Kutta Method, Fuzzy Cauchy Problem.

2. Preliminaries

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); \quad a \le t \le b, \\ y(a) = \alpha. \end{cases} \tag{1}$$

The basis of all Rung-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = h\sum_{i=1}^{m} w_i k_i . \tag{2}$$

where the w_i ' s for $i = 1,2,...m,$ are constants and

$$k_i = f(t_n + \alpha_i h, y_n + h\sum_{j=1}^{i-1} \beta_{ij} k_j). \tag{3}$$

Equation (2) is to be coincident whit Taylor series order $m.$ Therefore, the truncation error T_m , can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}).$$

The true magnitude of γ_m will generally be much less than the bound of Theorem 2.1. Thus, if the $O(h^{m+2})$ term is small compared with $\gamma_m h^{m+1},$ as we expect, to be so if h is small enough, then the bound on $\gamma_m h^{m+1},$ will usually be a bound on the error as a whole. The famous nonzero constants α_i , β_{ij} in 4th order Runge-Kutta method are

$$\alpha_1 = 0 , \alpha_2 = \alpha_3 = \tfrac{1}{2} , \alpha_4 = 1 , \beta_{21} = \tfrac{1}{2} , \beta_{32} = \tfrac{1}{2} , \beta_{43} = 1,$$

and we have, see [9]

$$\begin{aligned} &y_0 = \alpha , \\ &k_1 = f(t_i, y_i), \\ &k_2 = f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} k_1), \\ &k_3 = f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} k_2), \\ &k_4 = f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} k_3), \\ &y_{i+1} = y_i + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{aligned} \tag{4}$$

where

$$a = t_0 < t_1 < ... < t_N = b \text{ and } h = \frac{b-a}{N} = t_{i+1} - t_i . \tag{5}$$

Theorem 2.1. Let $f(t,y)$ belong to $C^4[a,b]$ and let its partial derivatives be bounded and assume there exist, P,M, positive numbers, such that

$$|f(t,y)|<M,\ |\frac{\partial^{i+j} f}{\partial t^i \partial y^j}|<\frac{P^{i+j}}{M^{j-1}},\quad i+j\le m,$$

then in the 4th order Rung-Kutta method (Proof [9]).

$$y(t_{i+1})-y_{i+1}=\tfrac{73}{720}h^5MP^4+O(h^6).$$

A triangular fuzzy number v, is defined by three numbers $a_1<a_2<a_3$ where the graph of $v(x)$, the membership function of the fuzzy number v, is a triangle with base on the interval $[a_1,a_3]$, and vertex at $x=a_2$. We specify v as $(a_1/a_2/a_3)$. We will write: (1)$v>0$ if $a_1>0$; (2)$v\ge 0$ if $a_1\ge 0$; (3)$v<0$ if $a_3<0$; and (4)$v\le 0$ if $a_3\le 0$

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level sets It means that if $v\in E$ then the r-level set

$$[v]_r=\{s\,|\,v(s)\ge r\},\quad 0<r\le 1,$$

is a closed bounded interval which is denoted by

$$[v]_r=[v_1(r),v_2(r)].$$

Let I be a real interval. A mapping $x:I\to E$ is called a fuzzy process and its r-level set is denoted by

$$[x(t)]_r=[x_1(t;r),x_2(t;r)],\quad t\in I,\quad r\in(0,1].$$

The derivative $x'(t)$ of a fuzzy process x is defined by

$$[x'(t)]_r=[x'_1(t;r),x'_2(t;r)],\quad t\in I,\quad r\in(0,1],$$

provided that this equation defines a fuzzy number, as in Sikkala [7].

Lemma 2.1. Let $v,w\in E$ and s be a scalar (.see [7]). Then for $r\in(0,1]$

$$\begin{aligned}
[v+w]_r &= [v_1(r)+w_1(r),v_2(r)+w_2(r)],\\
[v-w]_r &= [v_1(r)-w_2(r),v_2(r)-w_1(r)],\\
[v.w]_r &= [\min\{v_1(r).w_1(r),v_1(r).w_2(r),v_2(r).w_1(r),v_2(r).w_2(r)\},\\
&\qquad \max\{v_1(r).w_1(r),v_1(r).w_2(r),v_2(r).w_1(r),v_2(r).w_2(r)\}],\\
[s.v]_r &= s[v]_r.
\end{aligned}$$

3. A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I = [0, T], \\ y(0) = y_0, \end{cases} \tag{6}$$

where f is a continuous mapping from $R_+ \times R$ into R and $y_0 \in E$ with r-level sets

$$[y_0]_r = [y_1(0;r), y_2(0;r)], \quad r \in (0,1].$$

The extension principle of Zadeh leads to the following definition of $f(t, y)$ when $y = y(t)$ is a fuzzy number

$$f(t, y)(s) = \sup\{y(\tau) \mid s = f(t, \tau)\}, \quad s \in R.$$

It follows that

$$[f(t, y)]_r = [f_1(t, y; r), f_2(t, y; r)], \quad r \in (0,1],$$

where

$$\begin{aligned} f_1(t, y; r) &= \min\{f(t,u) \mid u \in [y_1(t;r), yx_2(t;r)]\}, \\ f_2(t, y; r) &= \max\{f(t,u) \mid u \in [y_1(t;r), y_2(t;r)]\}. \end{aligned} \tag{7}$$

Theorem 3.1. *Let f satisfy*

$$|f(t,v) - f(t,\bar{v})| \le g(t, |v - \bar{v}|), \quad t \ge 0, \quad v, \bar{v} \in R,$$

where $g : R_+ \times R_+ \to R_+$ is a continuous mapping such that $r \to g(t,r)$ is nondecreasing, the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \tag{8}$$

has a solution on R_+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (8) for $u_0 = 0$. Then the fuzzy initial value problem (6) has a unique fuzzy solution.

Proof [7].

4. 4th Order Runge-Kutta Method

Let the exact solution $[Y(t)]_r = [Y_1(t;r), Y_2(t;r)]$ be approximated by some $[y(t)]_r = [y_1(t;r), y_2(t;r)]$. From (2),(3) we define

$$
\begin{aligned}
y_1(t_{n+1};r) - y_1(t_n;r) &= h\sum_{i=1}^{4} w_i k_{i,1}(t_n, y(t_n;r)),\\
y_2(t_{n+1};r) - y_2(t_n;r) &= h\sum_{i=1}^{4} w_i k_{i,2}(t_n, y(t_n;r)),
\end{aligned} \tag{9}
$$

where the w_i's are constants and

$$
\begin{aligned}
&[k_i(t, y(t;r))]_r = [k_{i,1}(t, y(t;r)), k_{i,2}(t, y(t;r))],\ i = 1,2,3,4\\
&k_{i,1}(t_n, y(t_n;r)) = f(t_n + \alpha_i h\ ,\ y_1(t_n) + h\sum_{j=1}^{i-1}\beta_{i,j} k_{j,1}(t_n, y(t_n;r))),\\
&k_{i,2}(t_n, y(t_n;r)) = f(t_n + \alpha_i h\ ,\ y_2(t_n) + h\sum_{j=1}^{i-1}\beta_{i,j} k_{j,2}(t_n, y(t_n;r)))
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
k_{1,1}(t, y(t;r)) &= \min\{ f(t,u) \,|\, u \in [y_1(t;r), y_2(t;r)]\},\\
k_{1,2}(t, y(t;r)) &= \max\{ f(t,u) \,|\, u \in [y_1(t;r), y_2(t;r)]\},\\
k_{2,1}(t, y(t;r)) &= \min\{ f(t + \tfrac{h}{2}, u) \,|\, u \in [z_{1,1}(t; y(t;r)), z_{1,2}(t; y(t;r))]\},\\
k_{2,2}(t, y(t;r)) &= \max\{ f(t + \tfrac{h}{2}, u) \,|\, u \in [z_{1,1}(t; y(t;r)), z_{1,2}(t; y(t;r))]\},\\
k_{3,1}(t, y(t;r)) &= \min\{ f(t + \tfrac{h}{2}, u) \,|\, u \in [z_{2,1}(t; y(t;r)), z_{2,2}(t; y(t;r))]\},\\
k_{3,2}(t, y(t;r)) &= \max\{ f(t + \tfrac{h}{2}, u) \,|\, u \in [z_{2,1}(t; y(t;r)), z_{2,2}(t; y(t;r))]\},\\
k_{4,1}(t, y(t;r)) &= \min\{ f(t + h, u) \,|\, u \in [z_{3,1}(t; y(t;r)), z_{3,2}(t; y(t;r))]\},\\
k_{4,2}(t, y(t;r)) &= \max\{ f(t + h, u) \,|\, u \in [z_{3,1}(t; y(t;r)), z_{3,2}(t; y(t;r))]\}.
\end{aligned} \tag{11}
$$

Where in the 4th order Runge-Kutta method,

$$
\begin{aligned}
z_{1,1}(t, y(t;r)) &= y_1(t;r) + \tfrac{h}{2} k_{1,1}(t, y(t;r)) \quad , \quad z_{1,2}(t, y(t;r)) = y_2(t;r) + \tfrac{h}{2} k_{1,2}(t, y(t;r)),\\
z_{2,1}(t, y(t;r)) &= y_1(t;r) + \tfrac{h}{2} k_{2,1}(t, y(t;r)) \quad , \quad z_{2,2}(t, y(t;r)) = y_2(t;r) + \tfrac{h}{2} k_{2,2}(t, y(t;r)),\\
z_{3,1}(t, y(t;r)) &= y_1(t;r) + h k_{3,1}(t, y(t;r)) \quad , \quad z_{3,2}(t, y(t;r)) = y_1(t;r) + h k_{3,2}(t, y(t;r)).
\end{aligned} \tag{12}
$$

Define,

$$
\begin{aligned}
F[t, y(t;r)] &= k_{1,1}(t, y(t;r)) + 2k_{2,1}(t, y(t;r)) + 2k_{3,1}(t, y(t;r)) + k_{4,1}(t, y(t;r)),\\
G[t, y(t;r)] &= k_{1,2}(t, y(t;r)) + 2k_{2,2}(t, y(t;r)) + 2k_{3,2}(t, y(t;r)) + k_{4,2}(t, y(t;r)).
\end{aligned} \tag{13}
$$

The exact and approximate solutions at t_n, $0 \le n \le N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)]$ and $[y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$, respectively. The solution is calculated at grid points of (5). By (9),(13) we have

$$
\begin{aligned}
Y_1(t_{n+1};r) &\approx Y_1(t_n;r) + \tfrac{h}{6} F[t_n, Y(t_n); r],\\
Y_2(t_{n+1};r) &\approx Y_2(t_n;r) + \tfrac{h}{6} G[t_n, Y(t_n); r].
\end{aligned} \tag{14}
$$

We define

$$y_1(t_{n+1};r) = y_1(t_n;r) + \tfrac{h}{6} F[t_n, y(t_n); r],$$
$$y_2(t_{n+1};r) = y_2(t_n;r) + \tfrac{h}{6} G[t_n, y(t_n); r], \qquad (15)$$

the following lemmas will be applied to show convergence of these approximates,i.e.,

$$\lim_{h \to 0} y_1(t;r) = Y_1(t;r),$$
$$\lim_{h \to 0} y_2(t;r) = Y_2(t;r).$$

Lemma 4.1. *Le tthe sequence of numbers* $\{W_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \le A|W_n| + B, \quad 0 \le n \le N-1,$$

for some given positive constants A *and* B *(* Proof [10]).*Then*

$$|W_n| \le A^n |W_0| + B\frac{A^n - 1}{A - 1}, \quad 0 \le n \le N .$$

Lemma 4.2. Let the sequence of numbers $\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ satisfy

$$|W_{n+1}| \le |W_n| + A.\max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \le |V_n| + A.\max\{|W_n|, |V_n|\} + B,$$

for some given positive constants be A *and* B *., and denote*

$$|U_n| = |W_n| + |V_n|, \quad 0 \le n \le N .$$

Then

$$|U_n| \le \bar{A}^n |U_0| + \bar{B}\frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \le n \le N ,$$

where $\bar{A} = 1 + 2A$ *and* $\bar{B} = 2B$ *(* Proof [10]).

Let $F(t,u,v)$ and $G(t,u,v)$ be obtained by substituting $[y(t)]_r = [u,v]$ in (13).

$$F[t,u,v] = k_{1,1}(t,u,v) + 2k_{2,1}(t,u,v) + 2k_{3,1}(t,u,v) + k_{4,1}(t,u,v),$$
$$G[t,u,v] = k_{1,2}(t,u,v) + 2k_{2,2}(t,u,v) + 2k_{3,2}(t,u,v) + k_{4,2}(t,u,v).$$

The domain where F and G are defined is therefore

$$K = \{(t,u,v) \mid 0 \le t \le T, -\infty < v < \infty, -\infty < u \le v\} .$$

Theorem 4.1. *Let* $F(t,u,v)$ *and* $G(t,u,v)$ *belonged to* $C^4(K)$ *and let the partial derivatives of* F *and* G *be bounded over* K. *Then, for arbitrary fixed* $r: 0 \le r \le 1$, *the approximative solutions (14) converge to the exact solutions* $Y_1(t;r)$ *and* $Y_2(t;r)$ *uniformly in t.*

Proof. It is sufficient to show

$$\lim_{h\to 0} y_1(t_N;r) = Y_1(t_N;r),$$

$$\lim_{h\to 0} y_2(t_N;r) = Y_2(t_N;r),$$

where $t_N = T$. For $n = 0,1,..., N-1$, by using Taylor theorem we get

$$\begin{aligned} Y_1(t_{n+1};r) &= Y_1(t_n;r) + \tfrac{h}{6} F[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{73}{720} h^5 MP^4 + O(h^6), \\ Y_2(t_{n+1};r) &= Y_2(t_n;r) + \tfrac{h}{6} G[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{73}{720} h^5 MP^4 + O(h^6), \end{aligned} \tag{16}$$

denote

$$W_n = Y_1(t_n;r) - y_1(t_n;r),$$
$$V_n = Y_2(t_n;r) - y_2(t_n;r).$$

Hence from (15) and (16)

$$W_{n+1} = W_n + \tfrac{h}{6}\{F[t_n, Y_1(t_n;r), Y_2(t_n;r)] - F[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \tfrac{73}{720} h^5 MP^4 + O(h^6),$$

$$V_{n+1} = V_n + \tfrac{h}{6}\{G[t_n, Y_1(t_n;r), Y_2(t_n;r)] - G[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \tfrac{73}{720} h^5 MP^4 + O(h^6).$$

Then

$$|W_{n+1}| \le |W_n| + \tfrac{1}{3} Lh.\max\{|W_n|, |V_n|\} + \tfrac{73}{720} h^5 MP^4 + O(h^6),$$

$$|V_{n+1}| \le |V_n| + \tfrac{1}{3} Lh.\max\{|W_n|, |V_n|\} + \tfrac{73}{720} h^5 MP^4 + O(h^6),$$

for $t \in [0,T]$ and $L > 0$ is a bound for the partial derivatives of F and G. Thus by

lemma 4.2

$$|W_n| \le (1+\tfrac{2}{3}Lh)^n |U_0| + (\tfrac{73}{360} h^5 MP^4 + O(h^6)) \frac{(1+\tfrac{2}{3}Lh)^n - 1}{\tfrac{2}{3}Lh},$$

$$|V_n| \le (1+\tfrac{2}{3}Lh)^n |U_0| + (\tfrac{73}{360} h^5 MP^4 + O(h^6)) \frac{(1+\tfrac{2}{3}Lh)^n - 1}{\tfrac{2}{3}Lh},$$

where $|U_0|=|W_0|+|V_0|$. In particular

$$|W_N|\le(1+\tfrac{2}{3}Lh)^N|U_0|+(\tfrac{73}{240}h^4MP^4+O(h^5))\frac{(1+\frac{2}{3}Lh)^{\frac{T}{h}}-1}{L},$$
$$|V_N|\le(1+\tfrac{2}{3}Lh)^N|U_0|+(\tfrac{73}{240}h^4MP^4+O(h^5))\frac{(1+\frac{2}{3}Lh)^{\frac{T}{h}}-1}{L}.$$

Since $W_0=V_0=0$ we obtain

$$|W_N|\le(\tfrac{73}{240}MP^4)\frac{e^{\frac{2}{3}LT}-1}{L}h^4+O(h^5),$$
$$|V_N|\le(\tfrac{73}{240}MP^4)\frac{e^{\frac{2}{3}LT}-1}{L}h^4+O(h^5)$$

and if $h\to 0$ we get $W_N\to 0$ and $V_N\to 0$ which completes the proof.

5. Examples

Example 5.1. *Consider the fuzzy initial value problem, [10],*

$$\begin{cases} y'(t)=y(t),\ t\in I=[0,T], \\ y(0)=(0.75+0.25r,1.125-0.125r),\ 0<r\le 1. \end{cases}$$

By using 4th order Runge-Kutta method we have

$$y_1(t_{n+1};r)=y_1(t_n;r)[1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}],$$
$$y_2(t_{n+1};r)=y_2(t_n;r)[1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}],$$

The exact solution is given by

$$Y_1(t;r)=y_1(0;r)e^t \quad , \quad Y_2(t;r)=y_2(0;r)e^t$$

which at $t=1$,

$$Y(1;r)=[(0.75+0.25r)e,(1.125-0.125r)e],\ 0<r\le 1.$$

The exact and approximate solutions are compared and plotted at $t = 1$ in Figure (5.1).

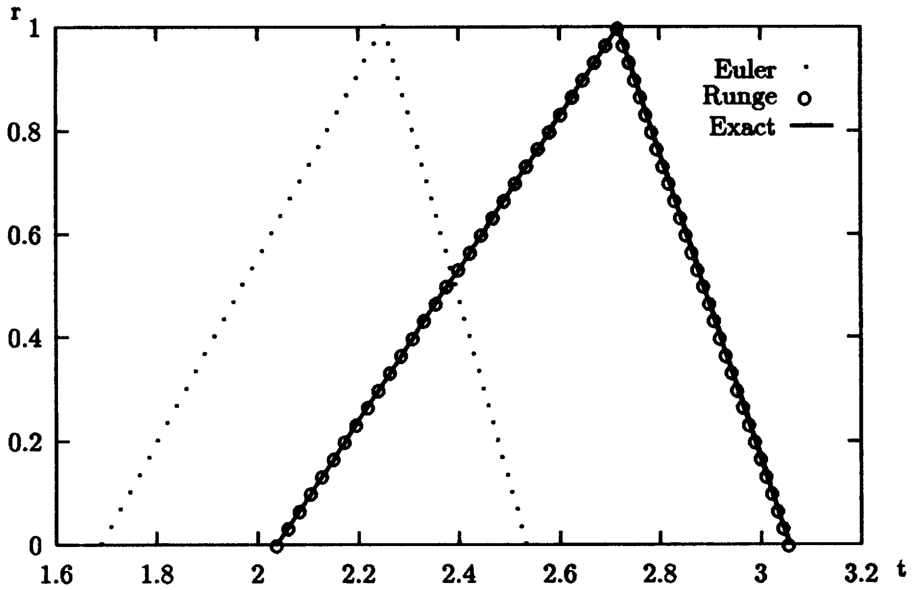

Figure (5.1), ($h=0.5$ **)**

Example 5.2. *Consider the fuzzy initial value problem, [10],*

$$\begin{cases} y'(t)=ty(t),\ [a,b]=[-1,1], \\ y(-1)=(\sqrt{e}-0.5(1-r),\sqrt{e}+0.5(1-r)),\ 0<r\le 1. \end{cases}$$

We separate between two steps.

$(a)\ t<0:$ The parametric form in this case is

$$y'_1(t;r)=t y_2(t;r) \quad , \quad y'_2(t;r)=t y_1(t;r),$$

with the initial conditions given. The unique exact solution is

$$Y_1(t;r)=\frac{A-B}{2}y_2(0;r)+\frac{A+B}{2}y_1(0;r),$$
$$Y_2(t;r)=\frac{A+B}{2}y_2(0;r)+\frac{A-B}{2}y_1(0;r),$$

where $A=e^{\frac{(t^2-a^2)}{2}}$, $B=\dfrac{1}{A}$.

$(b)\,t\ge 0:$ The parametric equations are

$$y'_1(t;r)=t y_1(t;r) \quad , \quad y'_2(t;r)=t y_2(t;r),$$

with the initial conditions given. The unique exact solution at $t > 0$ is

$$Y_1(t;r) = y_1(0;r)e^{\frac{t^2}{2}} \quad , \quad Y_2(t;r) = y_2(0;r)e^{\frac{t^2}{2}} .$$

By using 4th order Runge-Kutta method at $t_n, 0 \le n \le N$ we have

$$\begin{aligned}
k_{1,1}(t_n;r) &= \min\{t.u \mid u \in [y_1(t_n;r), y_2(t_n;r)]\},\\
k_{1,2}(t_n;r) &= \max\{t.u \mid u \in [y_1(t_n;r), y_2(t_n;r)]\},\\
k_{2,1}(t_n;r) &= \min\{(t+\tfrac{h}{2}).u \mid u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},\\
k_{2,2}(t_n;r) &= \max\{(t+\tfrac{h}{2}).u \mid u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},\\
k_{3,1}(t_n;r) &= \min\{(t+\tfrac{h}{2}).u \mid u \in [z_{2,1}(t_n;r), z_{2,2}(t_n;r)]\},\\
k_{3,2}(t_n;r) &= \max\{(t+\tfrac{h}{2}).u \mid u \in [z_{2,1}(t_n;r), z_{2,2}(t_n;r)]\},\\
k_{4,1}(t_n;r) &= \min\{(t+h).u \mid u \in [z_{3,1}(t_n;r), z_{3,2}(t_n;r)]\},\\
k_{4,2}(t_n;r) &= \max\{(t+h).u \mid u \in [z_{3,1}(t_n;r), z_{3,2}(t_n;r)]\}.
\end{aligned}$$

Where

$$\begin{aligned}
z_{1,1}(t_n;r) &= y_1(t_n;r) + \tfrac{h}{2}k_{1,1}(t_n;r) \quad , \quad z_{1,2}(t_n;r) = y_2(t_n;r) + \tfrac{h}{2}k_{1,2}(t_n;r),\\
z_{2,1}(t_n;r) &= y_1(t_n;r) + \tfrac{h}{2}k_{2,1}(t_n;r) \quad , \quad z_{2,2}(t_n;r) = y_2(t_n;r) + \tfrac{h}{2}k_{2,2}(t_n;r),\\
z_{3,1}(t_n;r) &= y_1(t_n;r) + hk_{3,1}(t_n;r) \quad , \quad z_{3,2}(t_n;r) = y_1(t_n;r) + hk_{3,2}(t_n;r).
\end{aligned}$$

By considering $t > 0$ and $t < 0$,the above minimizing and maximizing problems can be solved by GAMS software. The exact and approximate solutions are compared and plotted in Fig.(5.2).

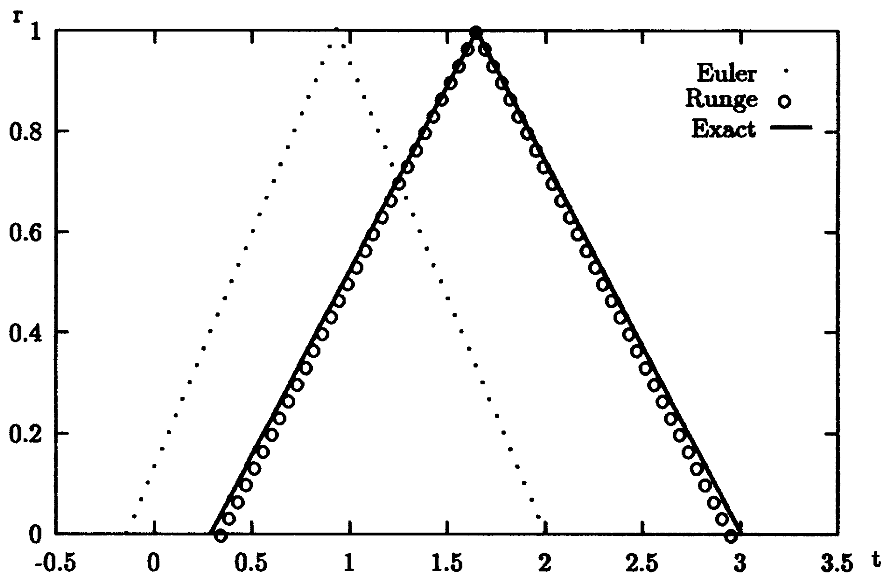

Figure (5.2), $(h = 0.4)$

Example 5.3. *Consider the fuzzy initial value problem*

$$y'(t) = c_1 y^2(t) + c_2 , \quad y(0) = 0,$$

where $c_i > 0$, *for* $i = 1,2$ *are triangular fuzzy numbers, [11].*

The exact solution is given by

$$Y_1(t;r) = l_1(r)\tan(\omega_1(r)\,t) ,$$

$$Y_2(t;r) = l_2(r)\tan(\omega_2(r)\,t) ,$$

with

$$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)} \quad , \quad l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)} ,$$

$$\omega_1(r) = \sqrt{c_{1,1}(r)/c_{2,1}(r)} \quad , \quad \omega_2(r) = \sqrt{c_{1,2}(r)/c_{2,2}(r)} ,$$

where

$$[c_1]_r = [c_{1,1}(r), c_{1,2}(r)] \quad \text{and} \quad [k_2]_r = [c_{2,1}(r), c_{2,2}(r)]$$

and

$$c_{1,1}(r) = 0.5 + 0.5r \; , \; c_{1,2}(r) = 1.5 - 0.5r \; , \; c_{2,1}(r) = 0.75 + 0.25r \; , \; c_{2,2}(r) = 1.25 - 0.25r.$$

The r-level sets of $y'(t)$ are

$$Y'_1(t;r) = c_{2,1}(r)\sec^2(\omega_1(r)\,t) ,$$

$$Y'_2(t;r) = c_{2,2}(r)\sec^2(\omega_2(r)\,t),$$

which defines a fuzzy number. We have

$$f_1(t,y;r) = \min\{c_1u^2 + c_2 \mid u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

$$f_2(t,y;r) = \max\{c_1u^2 + c_2 \mid u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

By using 4th order Runge-Kutta method at $t_n , 0 \le n \le N$ we have

$$k_{1,1}(t_n;r) = c_{1,1}(r).y_1^2(t_n;r) + c_{2,1}(r),$$

$$k_{1,2}(t_n;r) = c_{1,2}(r).y_2^2(t_n;r) + c_{2,2}(r),$$

$$k_{2,1}(t_n;r) = c_{1,1}(r).z_{1,1}^2(t_n;r) + c_{2,1}(r),$$

$$k_{2,2}(t_n;r) = c_{1,2}(r).z_{1,2}^2(t_n;r) + c_{2,2}(r),$$

$$k_{3,1}(t_n;r) = c_{1,1}(r).z_{2,1}^2(t_n;r) + c_{2,1}(r),$$

$$k_{3,2}(t_n;r) = c_{1,2}(r).z_{2,2}^2(t_n;r) + c_{2,2}(r),$$

$$k_{4,1}(t_n;r) = c_{1,1}(r).z_{3,1}^2(t_n;r) + c_{2,1}(r),$$

$$k_{4,2}(t_n;r) = c_{1,2}(r).z_{3,2}^2(t_n;r) + c_{2,2}(r).$$

where

$$z_{1,1}(t_n;r)=y_1(t_n;r)+\tfrac{h}{2}k_{1,1}(t_n;r) \quad , \quad z_{1,2}(t_n;r)=y_2(t_n;r)+\tfrac{h}{2}k_{1,2}(t_n;r),$$
$$z_{2,1}(t_n;r)=y_1(t_n;r)+\tfrac{h}{2}k_{2,1}(t_n;r) \quad , \quad z_{2,2}(t_n;r)=y_2(t_n;r)+\tfrac{h}{2}k_{2,2}(t_n;r),$$
$$z_{3,1}(t_n;r)=y_1(t_n;r)+hk_{3,1}(t_n;r) \quad , \quad z_{3,2}(t_n;r)=y_1(t_n;r)+hk_{3,2}(t_n;r).$$

There are two nonlinear programming problems which can be solved by GAMS software. Thus the suggested 4th order Runge-Kutta method of this paper can be used. The exact and approximate solutions are shown in Figure (5.3) at $t=1$.

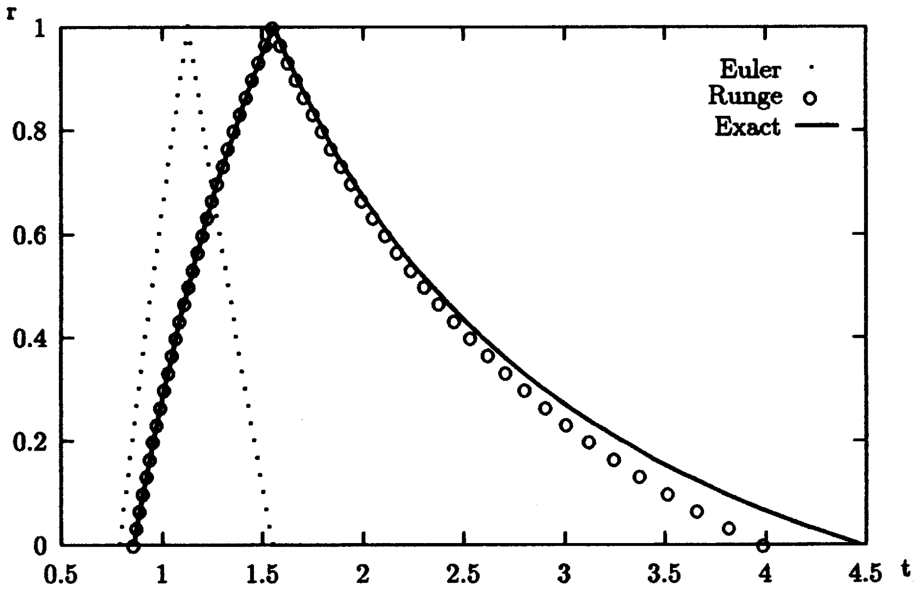

Figure (5.3), $(h=0.5)$

6. Conclusion

We note that the convergence order of the Euler method in [10] is $O(h)$. It is shown that in proposed method, the convergence order is $O(h^4)$ and the comparison of solutions of examples (1) ,(2) of this paper and [10] shows that these solutions are better for these examples.

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