A certain N-Generalized Principally Quasi-Baer Subring of the Matrix rings

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Abstract

For a fixed positive integer n, we say a ring with identity is *n-generalized right principally quasi-Baer*, if for any principal right ideal I of R, the right annihilator of I^n is generated by an idempotent. This class of rings includes the right principally quasi-Baer rings and hence all prime rings. A certain n-generalized principally quasi-Baer subring of the matrix ring $M_n(R)$ are studied, and connections to related classes of rings (e.g., p.q.-Baer rings and n-generalized p.p. rings) are considered[1].

1. Introduction and Preliminaries

Throughout all rings are assumed to be associative with identity. From [12, 21], a ring R is (quasi-)Baer if the right annihilator of any (right ideal) nonempty subset of R is generated, as a right ideal, by an idempotent. Moreover, in [12] Clark proved the left-right symmetry of this condition. He uses this condition to characterize when a finite dimensional algebra with unity over an algebraically closed filed is isomorphic to a twisted matrix units semigroup algebra. The class of quasi-Baer rings is a nontrivial generalization of the class of Baer rings. Every prime ring is quasi-Baer, hence prime rings with nonzero right singular ideal are quasi-Baer; but not Baer [24]. For a positive integer $n > 1$, the $n \times n$ matrix ring over a non-Prüfer commutative domain is a prime quasi-Baer ring which is not a Baer ring by [27] and [21, p.17]. The $n \times n$ $(n > 1)$ upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not

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Baer by an example due to Cohn; see [1], [20] and [5]. The theory of Baer and quasi-Baer rings has come to play an important role and major contributions have been made in recent years by a number of authors, including Birkenmeier, Chatters, Khuri, Kim, Hirano and Park (see, for example [1], [4-7], [16], [21], [26] and [28]).

A ring satisfying a generalization of *Rickart's condition* [30] (i.e., right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right p.p.-ring. A ring R is called a *right* (resp. *left) p.p.-ring* if every principal right (resp. left) ideal is projective. R is called a *p.p.-ring* (also called a Rickart ring [2, p.18]), if it is both right and left p.p.-ring. In [9] Chase shows the concept of p.p.-ring is not left-right symmetric. Small [30] shows that a right p.p.-ring R is Baer (so p.p), when R is orthogonally finite. Also it is shown by Endo [13] that a right p.p.-ring R is p.p when R is abelian (i.e., every idempotent is central). Finally Chatters and Xue [11] prove that in a duo (i.e., every one sided ideal is two sided) p.p.-ring R, if I is a finitely generated right projective ideal of R, then I is left projective and a direct summand of an invertible ideal. Following Birkenmeier et al. [7], R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*), if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, R is right p.q.-Baer if R modulo the right annihilator of any principal right ideal is projective. Similarly, left p.q.-Baer rings can be defined. If R is both right and left p.q.-Baer, then it is called p.q.-Baer. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, and all abelian p.p.-rings. A ring R is said to be $p-$regular, if for every $x \in R$ there exists a natural number n, depending on x, such that $x^n \in x^n R x^n$. A ring R is called a *generalized right p.p.-ring* if for any $x \in R$ the right ideal $x^n R$ is projective for some positive integer n, depending on x, or equivalently, if for any $x \in R$ the right annihilator of x^n is generated by an idempotent for some positive integer n, depending on x. A ring is called *generalized p.p.-ring*, if it is both generalized right and left p.p.-ring.

Note that Von Neumann regular rings are right (left) p.p.-rings by Goodearl [14, Theorem 1.1], and a same argument as [14, Theorem 1.1] shows that p – regular rings are generalized p.p.-rings. Right p.p.-rings are generalized right p.p obviously. See [18] for more details.

Definition 1.1. Given a fixed positive integer n, we say a ring R is n *-generalized right principally quasi Baer* (or n-generalized right p.q.-Baer), if for all principal right ideal I of R, the right annihilator of I^n is generated by an idempotent. Left cases may be defined analogously.

The class of n-generalized right p.q.-Baer rings includes all right p.q.-Baer rings, (and hence all biregular rings, quasi-Baer rings, abelian p.p.-rings and *semicommutative* (i.e., if $r(x)$ is an ideal for all $x \in R$) generalized p.p rings). Theorem 2.1 in section 2, allows us to construct examples of n-generalized p.q.- Baer rings that are not p.q.-Baer. Some conditions on the equivalence of n-generalized p.q.-Baer and n-generalized p.p.-rings are discussed. However, we show by examples that the class of n-generalized p.q.-Baer rings properly extends the aforementioned classes.

In this paper, we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. We study n-generalized p.q.-Baer subrings of the matrix ring $M_n(R)$. Theorem 2.2, enables us to generate examples of n-generalized p.q.-Baer subrings of the matrix ring $M_n(R)$. Theorem 2.2, which extends [18, Proposition 6], enables us to provide more examples of matrix rings, that are both n-generalized p.q.-Baer and n-generalized p.p.-ring. Connections to related classes of rings are investigated. Although the class of n-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, and all abelian p.p. rings), however we show by examples that the class of n-generalized p.q.-Baer rings properly extends the aforementioned classes.

Note that, for a *reduced* ring (which has no nonzero nilpotent elements), we have $l_R(Rx) = l_R((Rx)^n) = l_R(x^n) = l_R(x) = r_R(x) = r_R(x^n) = r_R((xR)^n) = r_R(xR)$, for every $x \in R$ and every positive integer n. Therefore reduced rings are semicommutative and semicommutative rings are abelian. Also for reduced rings the definitions of right p.q.-

Baer, n-generalized right p.q.-Baer, generalized p.p. and p.p.-ring are coincide. This leads one ask whether commutative reduced rings are n-generalized p.q-Baer. However, the answer is negative by the following.

Example 1.2. Let p be a prime number and $R = \{(a,b) \in Z \bigoplus Z \mid a \equiv b \ (mod p)\}$, then R is a commutative reduced ring. Note that the only idempotents of R are $(0,0)$ and $(1,1)$. One can show that $r_R((p,0)R) = (0,p)R$, so $r_R((p,0)R)$ dose not contain a nonzero idempotent of R; and hence R is not n-generalized right quasi-Baer, for any positive integer n.

Lemma1.3. Let R be an abelian n-generalized right p.q.-Baer ring, then $r_R(I^n) = r_R(I^m)$ for every principal right ideal I of R and each positive integer m with $n \le m$.

Proof. It is enough to show that $r_R(I^n) = r_R(I^{n+1})$. Let $x \in r_R(I^{n+1})$, then $Ix \subseteq r_R(I^n) = fR$ for some idempotent $f \in R$. Hence $I^n x = I^n xf = 0$. Thus $x \in r_R(I^n)$.

2. N -generalized right principally quasi Baer subrings of the matrix rings

In this section we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. Theorem 2.3, which extends [18, Proposition 6], enables us to provide more examples of matrix rings that are both n-generalized p.q.-Baer and n-generalized p.p.-ring. We begin with Theorem 2.2 below, which enables us to generate examples of n-generalized p.q.-Baer subrings of the matrix ring $M_n(R)$:

Lemma 2.1[18, Lemma 2]. Let R be an abelian ring and define

$$S_n := \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\},$$

with n a positive integer ≥ 2. Then every idempotent in S_n is of the form

$$\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix} \text{with } f^2 = f \in R.$$

We will use S_n Throughout the remainder of the paper, to denote the ring indicated in Lemma 2.1.

Theorem 2.2. If R is an abelian p.q.-Baer ring and n (≥ 2) is a positive integer, then S_n is an n-generalized right p.q.-Baer ring.

Proof. We proceed by induction on n. It is easy to show that S_2 is a 2-generalized right p.q.-Baer ring. Let I_n be a principal right ideal of S_n. Consider $I_{n-1,1} = \{B \in S_{n-1} \mid B$ is obtained by deleting n-th row and n-th column of a matrix in $I_n\}$, and $I_{n-1,2} = \{B \in S_{n-1} \mid B$ is obtained by deleting 1-th row and 1-th column of a matrix in $I_n\}$. It is clear that $I_{n-1,1}$ and $I_{n-1,2}$ are principal right ideals of S_{n-1}. By induction hypothesis and Lemma 2.1, there are $e_i^2 = e_i \in S_{n-1}, f_i^2 = f_i \in R$ for $i = 1,2$ such that $r_{S_{n-1}}(I_{n-1,i}^{n-1}) = e_i S_{n-1}$, $e_i = \begin{pmatrix} f_i & 0 & \cdots & 0 \\ 0 & f_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_i \end{pmatrix}$. Let J be the set of entries of the main diagonal of the elements of $I_{n-1,1}$ or $I_{n-1,2}$. It is clear that J is a principal right ideal of R. Since R is right p.q.-Baer, $r_R(J) = f_1 R = f_2 R$. Hence $f_1 = f_2$, since R is an abelian ring. Now let

$$X = \begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix} \in r_{S_n}(I_n^n) \text{ and } Y = \begin{pmatrix} a_1a_2a_3\cdots a_n & y_{12} & \cdots & y_{1n} \\ 0 & a_1a_2a_3\cdots a_n & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1a_2a_3\cdots a_n \end{pmatrix} \in I_n^n$$

Since $r_{S_{n-1}}(I_{n-1,1}^{n-1}) = r_{S_{n-1}}(I_{n-1,2}^{n-1}) = e_1 S_{n-1}$, x and x_{ij} 's are in $f_1 R$ for each i and j except x_{1n}. So we have $a_1a_2\cdots a_n x_{1n} + y_{1n}x = 0$. Hence $y_{1n}x = 0$, since $f_1 \in B(R)$. Thus $x_{1n} \in f_1 R$ and hence $r_{S_n}(I_n^n) \subseteq eS_n$ for

$$e = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_1 \end{pmatrix} \in S_n .$$

Since, for each $Y \in I_n e$, all entries of the main diagonal of Y are zero and e is central, $I_n^n e = (I_n e)^n = 0$. Thus $r_{S_n}(I_n^n) = eS_n$. Therefore S_n is n-generalized right p.q.-Baer.

The following result, which generalizes [18, Proposition 6], provides examples of matrix rings that are both n-generalized p.q.-Baer and n-generalized p.p.-ring:

Theorem 2.3. If R is an abelian p.p.-ring, then S_n is an abelian n-generalized p.p.-ring.

Proof. We prove by induction on n. First, we show that the trivial extension S_2 of R is 2-generalized right p.p. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in S_2$ and $r_R(a) = eR$, with $e = e^2 \in R$. It is clear that, $fR \subseteq r_{S_2}(A^2)$ with $f = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$. Next, let $A^2 \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = 0$. Since R is reduced, $a^2x = ax = 0$ and $a^2y = ay = 0$. Hence $ex = x$ and $y = ey$. Thus $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = f\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$.

Therefore S_2 is 2-generalized right p.p. Now assume $B = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in S_n$.

Consider $B_1 = \begin{pmatrix} a & a_{12} & \cdots & a_{1n-1} \\ 0 & a & \cdots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$ and $B_2 = \begin{pmatrix} a & a_{23} & \cdots & a_{2n} \\ 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$ in S_{n-1}, then by the induction hypothesis, there exists $e_i^2 = e_i \in S_{n-1}$, $f_i^2 = f_i \in R$, such that $r_{S_{n-1}}(B_i^{n-1}) = e_iS_{n-1}$, $e_i = \begin{pmatrix} f_i & 0 & \cdots & 0 \\ 0 & f_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_i \end{pmatrix}$ for $i = 1,2$. By direct calculations, we have $r_{S_n}(B^{2n-2}) = eS_n$ with $e = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$. Since $r_R(a) = fR$, by [27, Lemma 3], $r_{S_n}(B^n) = r_{S_n}(B^{2n-2}) = eS_n$.

Corollary 2.4 [18, Proposition 6]. If R is a domain, then S_n is an abelian n-generalized p.p.-ring.

For a semicommutative ring, the definitions of n-generalized right p.q.-Baer and n-generalized right p.p. are coincide:

Proposition 2.5. Let R be a semicommutative ring. Then R is n-generalized right p.q.-Baer if and only if R is n-generalized right p.p.

Proof. Let R be n-generalized right p.q.-Baer and $a \in R$. Then $r_R(aR)^n = eR$ for some idempotent $e \in R$. Let $x \in r_R(a^n)$. Since R is semicommutative, $RaRx \subseteq r_R(a^{n-1})$, which implies that $r_R(aR)^n = eR$. The converse is similar.

There exists an n-generalized right p.q.-Baer ring, which is generalized p.p.-ring but is not semicommutative.

Example 2.6. Let R be an integral domain and S_4 be defined over R. Then S_4 is abelian 4-generalized p.p.-ring and is 4-generalized p.q.-Baer by Corollary 2.4. By considering $b = a = e_{12} + e_{14} + e_{34}$ and $c = e_{23}$ in S_4, where e_{ij} denote the matrix units, we have $ab = 0$, and $acb \neq 0$, hence $aS_4b \neq 0$.

Now we conjecture that subrings of n-generalized right p.q.-Baer rings are also n-generalized right p.q.-Baer. But the answer is negative by the following.

Example 2.7. For a field F, take $F_n = F$ for $n = 1, 2, \ldots,$ and let S be the 2×2 matrix ring over the ring $\Pi_{n=1}^{\infty} F_n$. By [7, Proposition 2.1 and Theorem 2.2] we have that S is a p.q.-Baer ring. Let

$$R = \begin{pmatrix} \Pi_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & < \bigoplus_{n=1}^{\infty} F_n, 1 > \end{pmatrix},$$

which is a subring of S, where $< \bigoplus_{n=1}^{\infty} F_n, 1 >$ is the F-algebra generated by $\bigoplus_{n=1}^{\infty} F_n$ and 1. Then by [7, Example 1.6], R is semiprime p.p which is neither right p.q.-Baer (and hence not n-generalized right p.q.-Baer), nor left p.q.-Baer (and hence not n-generalized left p.q.-Baer).

3. Examples of n-generalized p.q.-Baer subrings

Although the class of n-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, all quasi-Baer rings, and all abelian p.p. rings), however we show by examples that the class of n-generalized p.q.-Baer rings properly extends the aforementioned classes.

By the following example, there is an abelian p.q.-Baer (hence semiprime) ring R, which is not reduced, but S_n is an abelian n-generalized right p.q.-Baer ring that is not semiprime.

Example 3.1. By Zalesskii and Neroslavskii [10, Example 14.17, p.179], there is a simple noetherian ring R that is not a domain and in which 0 and 1 are the only idempotents. Thus R is an abelian p.q.-Baer ring that is neither left nor right p.p, and hence is not reduced. By [7, Proposition 1.17] R is semiprime and by Theorem 2.1, S_n is abelian n-generalized p.q.-Baer, that is not semiprime and hence is not right p.q.-Baer.

Example 3.2. If R is an abelian p.q-Baer ring, then $R[x]/ < x^3 >$ is an n-generalized p.q.-Baer ring.

Proof. First we note that $\Theta : T \to R[x]/ < x^3 >$ defined by

$$(a_0, a_1, a_2) \to (a_0 + a_1 x + a_2 x^2) + < x^3 >$$

is an isomorphism, where $T = \{(a,b,c) \mid a,b,c \in R\}$ is a ring with addition componentwise and the multiplication defined by

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 a_2, a_1 b_2 + b_1 a_2, a_1 c_2 + b_1 b_2 + c_1 a_2).$$

Let J be an ideal of T. Suppose $I = \{a \in R \mid (a,b,c) \in J\}$, it is clear that I is an ideal of R. Since R is p.q.-Baer, $r_R(I) = eR$ for an idempotent $e \in R$. We can show that $r(J^3) = (e,0,0)T$, and hence, the result follows.

There exists a commutative n-generalized p.q.-Baer (hence n-generalized p.p.-) ring R, over which S_n is not an n-generalized p.p.-ring.

Example 3.3. Let $p \neq 3$ be a prime integer and Z_{p^3} be the ring of integers modulo p^3, and S_3 be defined over Z_{p^3}. Let $A = pI_3 + e_{13}$, where I_3 is the identity matrix and e_{ij} denote the matrix units. It is clear that $pI_3 + e_{13} + e_{12} \in r_{S_n}(A^3)$ and idempotents of S_3 are I_3 and 0. Hence $r_{S_3}(A^3) \neq I_3 S_3$ and that S_3 is not 3-generalized p.p.-ring, but Z_{p^3} is a 3-generalized p.p.-ring.

Example 3.4. For every abelian quasi-Baer (resp. p.p.-) ring R, by Theorems 2.1 and

2.2, the ring S_n is n-generalized right p.q.-Baer, which is not right p.q.-Baer. Therefore we are able to provide examples of n-generalized right p.q.-Baer rings that is not right p.q.-Baer:

Let F be a field, and $R = F[x]$ be the polynomial ring where x is an indeterminate. Then S_n is a n-generalized right p.q.-Baer ring that is not right p.q.-Baer.

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