Contractibility and idempotents in Banach algebras

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Abstract

Let A be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of b-contractibility and one of its equivalent forms are introduced. Throgh an example, it is shown that b-contractibility is strictly weaker than contractibility.

Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4],[5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call b-contractibility. We give a characterization of b-contractibility analog to that of contractibility given by Taylor. Also, we show that b-contractibility is strictly weaker than contractibility.

2000 Mathematics Subject Classification, 46H10, 46H20.

Key words:Contractible Banach algebras, Minimal idempotent.

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First we recall some terminology. Throughout this paper, A is a Banach algebra and A-module means Banach A-bimodule. For a subset E of A, E' is the commutant of E. If for every A-bimodule X every bounded derivation from A into X is inner, then A is called *contractible*. Also, the term "semisimple" means *Jacobson semisimple*. An idempotent $e \in A$ is called *minimial* if eAe is a division ring. If e and f are idempotents in A, we write $e \le f$ if $fe = ef = e$ holds. A nonzero idempotent $e \in A$ is called *primitive* if $0 \le f \le e$ imlies that $f = 0$ or $f = e$. Also, two idempotents e and f are said to be *orthogonal* if they satisfy $ef = fe = 0$. Let S be a subset of A. The *right annihilator* of S in A which we denote by $ran(S)$ is the set

$$ran(S) = \{a \in A : ba = 0 \text{ for } b \in S\}.$$

The left annihilator $lan(S)$ is defined semilarly. The *annihilator* of S is the set $Ann(S) = ran(S) \cap lan(S)$.

Contractibility

Theorem 2.1. Let A be a contractible Banach algebra which is an ideal in a Banach algebra B. Then $A + A' = B$.

Proof. If $A + A' \neq B$, then we can choose $b \in B - (A + A')$. Now define

$$D : A \to A, x \mapsto xb - bx.$$

Clearly D is a derivation on A. By assumption there exists an $a \in A$ such that $D(x) = xa - ax$ for all $x \in A$. The latter result implies that $b - a \in A'$ or equivalently $b \in A + A'$ which contradicts the selection of b. Therefore $A + A' = B$.

Theorem 2.2. Let A be a contractible Banach algebra which is an ideal in a Banach algebra B. Then $B = A \oplus \mathrm{Ann}(A)$.

Proof. Since A is contractible then $M_2(A)$ with l^1-norm is contarctible, where $M_2(A)$ is the algebra of 2×2 matrices with the enteries from A. On the other hand $M_2(A)$ is an ideal in $M_2(B)$ and by Theorem 2.1 we have the equality $M_2(B) = M_2(A) + M_2(A)'$. One can easily observe that

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$$M_2(A)' = \begin{bmatrix} A' & Ann(A) \\ Ann(A) & A' \end{bmatrix}.$$

Thus $B = A + Ann(A)$. But $A \cap Ann(A) = 0$, because A is unital. Therefore the identity $B = A \oplus Ann(A)$ holds.

Remark. In Theorems 2.1 and 2.2, A and B are related only algebrically. Indeed if there exists an infinite dimensional contractible Banach algebra A which is an ideal in a Banach algebra B, then the norm topology of A could be different from the relative norm topology of A which inherits from B.

Theorem 2.3. Let A be a contractible Banach algebra which admits a nonzero multiplicative linear functional f. Then A contains a central minimal idempotent.

Proof. Let $d = \sum_{n=1}^{\infty} a_n \otimes b_n$ be a diagonal for A and define

$$T : A, \to a \mapsto \sum_{n=1}^{\infty} < f, aa_n > b_n.$$

Since $\sum_n a_n b_n = 1$, then

$$< f, T(1) > = < f, \sum_n < f, a_n > b_n > = \sum_n < f, a_n >< f, b_n >$$
$$= \sum_n < f, a_n b_n > = < f, \sum_n a_n b_n > = < f, 1 > = 1.$$

Thus T(1) $\neq 0$. Moreover for every $a \in A$ and $g, h \in A^*$ we have

$$< h, \sum_n < g, aa_n > b_n > = \sum_n < g, aa_n >< h, b_n >$$
$$= < g \otimes h, \sum_n aa_n \otimes b_n >$$
$$= < g \otimes h, \sum_n a_n \otimes b_n a >$$
$$= \sum_n < g, a_n >< h, b_n a >$$
$$= < h, \sum_n < g, a_n > b_n a > .$$

This implies that

$$\sum_n < g, aa_n > b_n = \sum_n < g, a_n > b_n a.$$

Thus we assume that

T(1)=e, then we have $T(a) = \sum_n < f, aa_n > b_n = \sum_n < f, a_n > b_n a = ea.$ On the other hand we have $T(a) = \sum_n < f, aa_n > b_n = < f, a > \sum_n < f, a_n > b_n = < f, a > e.$ Hence T is an operator of rank one and $e^2 = T(e) = < f, e > e = e.$ Now define

$$T_1 : A \to A, a \mapsto \sum_n a_n < f, aa_n >.$$

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With a similar argument we can show that

$$T_1(a) = ae' = \langle f, a \rangle e' \quad a \in A$$

where $e' = T_1(1)$. Also we have $e'^2 = e'$ and $\langle f, e' \rangle = 1$. Now the identities

$$ee' = \langle f, e' \rangle e = e, \qquad ee' = \langle f, e \rangle e' = e'$$

imply that $e = e'$ and for every $a \in A$ we have

$$ea = \langle f, a \rangle e = \langle f, a \rangle e' = ae' = ae.$$

Therefore e is a central idempotent. In addition since T is a rank one operator and $ranT = eAe$, then $eA = eAe = Ce$ is a division ring. Therefore e is a minimal idempotent.

b-Contractibility

Definition. Let A be a Banach algebra and π be the natural map,

$$\pi : A \otimes A \longrightarrow A, \qquad \pi(\sum_n a_n \otimes b_n) \to \sum_n a_n b_n .$$

Let $b \in A$ and X be an A-module. We say that a derivation $\delta A \longrightarrow X$ is a *b-derivation* if there exists another derivation $\delta' A \longrightarrow X$ such that $\delta = b\delta'$, where $(b\delta')(a) = b\delta'(a)$. Also we say that A is *b-contractible* if for every A-module X, every bounded b-derivation from A into X is inner. We call $d \in A \hat{\otimes} A$ a *b-diagonal* if $\pi(d) = b$ and $a.d = d.a$ for all $a \in A$.

Theorem 3.1. Let A be a unital Banach algebra and $b \in A' - \{0\}$. Then A is b-contractible if and only if A has a b-diagonal.

Proof. First suppose A is b-contractible and π is defined as above. Clearly $\ker\pi$ is an A-module and if we define

$$\delta : A \to \ker\pi, a \mapsto ab \otimes 1 - b \otimes a$$

then it is easy to see that δ is a b-derivation. Indeed $\delta = b\delta'$ where

$$\delta' : A \to \ker\pi, a \mapsto a \otimes 1 - 1 \otimes a$$

ince A is b-contractible, then threre exists an element $\sum_n c_n \otimes d_n \in \ker\pi$ such that

$$\delta(a) = \sum_n ac_n \otimes d_n - \sum_n c_n \otimes d_n a \quad a \in A.$$

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Let $d=b\otimes 1-\sum_n c_n\otimes d_n$. The above identities show that $\pi(d)=b$ and $a.d=d.a$ for all $a\in A$. Therefore, d is a b-diagonal for A.

Conversely suppose $d=\sum_n a_n\otimes b_n$ is a b-diagonal for A, X is an A-module and $\delta : A\longrightarrow X$ is a bounded derivation. Clearly the map

$$\psi : A\times A\to X,(a,c)\mapsto a\delta(c)$$

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map $\varphi : A\hat{\otimes} A\longrightarrow X$ such that $\varphi\circ\otimes=\psi$ that is $\varphi(a\otimes c)=a\delta(c)$. In particular using the fact that d is a b-diagonal for A, we get

$$\sum_n aa_n\delta(b_n)=\varphi(a.d)=\varphi(d.a)=\sum_n a_n\delta(b_na),\quad a\in A.$$

Now if $x=\sum_n a_n\delta(b_n)$, then for every $a\in A$ we have

$$ax-xa=\sum_n aa_n\delta(b_n)-\sum_n a_n\delta(b_n)a=\sum_n aa_n\delta(b_n)+b\delta(a)-\sum_n a_n\delta(b_na).$$

Thus the identity $ax-xa=b\delta(a)$ holds for every $a\in A$. Therefore every b-derivation is inner.

Example 3.2. Let A be the Banach algebra $l_1(N)$ with pointwise multiplication and $\{e_n\}$ be the standard basis for A.Then for every positive integer n, A is e_n-contractible. Indeed $e_n\otimes e_n$ is an e_n-diagonal for A. But A is not contractible, since it is not unital. Therefore b-contractibility dose not imply contractibility.

Remark. If A is contractible, then it is unital and one can easily observe that A is b-contractible for every $b\in A-\{0\}$. However the above example shows that for non-unital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

Problem. Does there exist a unital Banach algebra which is b-contactible for some nonzero central idempotent b, but is not contractible?

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