

Contractibility and idempotents in Banach algebras

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Abstract

Let A be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of b-contractibility and one of its equivalent forms are introduced. Through an example, it is shown that b-contractibility is strictly weaker than contractibility.

Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4], [5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call b-contractibility. We give a characterization of b-contractibility analog to that of contractibility given by Taylor. Also, we show that b-contractibility is strictly weaker than contractibility.

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First we recall some terminology. Throughout this paper, A is a Banach algebra and A -module means Banach A -bimodule. For a subset E of A , E' is the commutant of E . If for every A -bimodule X every bounded derivation from A into X is inner, then A is called *contractible*. Also, the term "semisimple" means *Jacobson semisimple*. An idempotent $e \in A$ is called *minimal* if eAe is a division ring. If e and f are idempotents in A , we write $e \leq f$ if $fe = ef = e$ holds. A nonzero idempotent $e \in A$ is called *primitive* if $0 \leq f \leq e$ implies that $f = 0$ or $f = e$. Also, two idempotents e and f are said to be *orthogonal* if they satisfy $ef = fe = 0$. Let S be a subset of A . The *right annihilator* of S in A which we denote by $\text{ran}(S)$ is the set

$$\text{ran}(S) = \{a \in A : ba = 0 \text{ for } b \in S\}.$$

The left annihilator $\text{lan}(S)$ is defined similarly. The *annihilator* of S is the set $\text{Ann}(S) = \text{ran}(S) \cap \text{lan}(S)$.

Contractibility

Theorem 2.1. Let A be a contractible Banach algebra which is an ideal in a Banach algebra B . Then $A + A' = B$.

Proof. If $A + A' \neq B$, then we can choose $b \in B - (A + A')$. Now define

$$D : A \rightarrow A, x \mapsto xb - bx.$$

Clearly D is a derivation on A . By assumption there exists an $a \in A$ such that $D(x) = xa - ax$ for all $x \in A$. The latter result implies that $b - a \in A'$ or equivalently $b \in A + A'$ which contradicts the selection of b . Therefore $A + A' = B$.

Theorem 2.2. Let A be a contractible Banach algebra which is an ideal in a Banach algebra B . Then $B = A \oplus \text{Ann}(A)$.

Proof. Since A is contractible then $M_2(A)$ with l^1 -norm is contractible, where $M_2(A)$ is the algebra of 2×2 matrices with the entries from A . On the other hand $M_2(A)$ is an ideal in $M_2(B)$ and by Theorem 2.1 we have the equality $M_2(B) = M_2(A) + M_2(A)'$. One can easily observe that

$$M_2(A)' = \begin{bmatrix} A' & Ann(A) \\ Ann(A) & A' \end{bmatrix}.$$

Thus $B = A + Ann(A)$. But $A \cap Ann(A) = 0$, because A is unital. Therefore the identity $B = A \oplus Ann(A)$ holds.

Remark. In Theorems 2.1 and 2.2, A and B are related only algebraically. Indeed if there exists an infinite dimensional contractible Banach algebra A which is an ideal in a Banach algebra B , then the norm topology of A could be different from the relative norm topology of A which inherits from B .

Theorem 2.3. Let A be a contractible Banach algebra which admits a nonzero multiplicative linear functional f . Then A contains a central minimal idempotent.

Proof. Let $d = \sum_{n=1}^{\infty} a_n \otimes b_n$ be a diagonal for A and define

$$T : A \rightarrow A \mapsto \sum_{n=1}^{\infty} \langle f, aa_n \rangle b_n.$$

Since $\sum_n a_n b_n = 1$, then

$$\begin{aligned} \langle f, T(1) \rangle &= \langle f, \sum_n \langle f, a_n \rangle b_n \rangle = \sum_n \langle f, a_n \rangle \langle f, b_n \rangle \\ &= \sum_n \langle f, a_n b_n \rangle = \langle f, \sum_n a_n b_n \rangle = \langle f, 1 \rangle = 1. \end{aligned}$$

Thus $T(1) \neq 0$. Moreover for every $a \in A$ and $g, h \in A^*$ we have

$$\begin{aligned} \langle h, \sum_n \langle g, aa_n \rangle b_n \rangle &= \sum_n \langle g, aa_n \rangle \langle h, b_n \rangle \\ &= \langle g \otimes h, \sum_n aa_n \otimes b_n \rangle \\ &= \langle g \otimes h, \sum_n a_n \otimes b_n a \rangle \\ &= \sum_n \langle g, a_n \rangle \langle h, b_n a \rangle \\ &= \langle h, \sum_n \langle g, a_n \rangle b_n a \rangle. \end{aligned}$$

This implies that

$$\sum_n \langle g, aa_n \rangle b_n = \sum_n \langle g, a_n \rangle b_n a.$$

Thus we assume that

$T(1) = e$, then we have $T(a) = \sum_n \langle f, aa_n \rangle b_n = \sum_n \langle f, a_n \rangle b_n a = ea$. On the other hand we have $T(a) = \sum_n \langle f, aa_n \rangle b_n = \langle f, a \rangle \sum_n \langle f, a_n \rangle b_n = \langle f, a \rangle e$. Hence T is an operator of rank one and $e^2 = T(e) = \langle f, e \rangle e = e$. Now define

$$T_1 : A \rightarrow A, a \mapsto \sum_n a_n \langle f, aa_n \rangle.$$

With a similar argument we can show that

$$T_1(a) = ae' = \langle f, a \rangle e' \quad a \in A$$

where $e' = T_1(1)$. Also we have $e'^2 = e'$ and $\langle f, e' \rangle = 1$. Now the identities

$$ee' = \langle f, e' \rangle e = e, \quad ee' = \langle f, e \rangle e' = e'$$

imply that $e = e'$ and for every $a \in A$ we have

$$ea = \langle f, a \rangle e = \langle f, a \rangle e' = ae' = ae.$$

Therefore e is a central idempotent. In addition since T is a rank one operator and $\text{ran} T = eAe$, then $eA = eAe = Ce$ is a division ring. Therefore e is a minimal idempotent.

b-Contractibility

Definition. Let A be a Banach algebra and π be the natural map,

$$\pi : A \otimes A \longrightarrow A, \quad \pi \left(\sum_n a_n \otimes b_n \right) \rightarrow \sum_n a_n b_n.$$

Let $b \in A$ and X be an A -module. We say that a derivation $\delta : A \longrightarrow X$ is a *b-derivation* if there exists another derivation $\delta' : A \longrightarrow X$ such that $\delta = b\delta'$, where $(b\delta')(a) = b\delta'(a)$. Also we say that A is *b-contractible* if for every A -module X , every bounded b -derivation from A into X is inner. We call $d \in \hat{A} \otimes A$ a *b-diagonal* if $\pi(d) = b$ and $a.d = d.a$ for all $a \in A$.

Theorem 3.1. Let A be a unital Banach algebra and $b \in A' - \{0\}$. Then A is *b-contractible* if and only if A has a *b-diagonal*.

Proof. First suppose A is *b-contractible* and π is defined as above. Clearly $\ker \pi$ is an A -module and if we define

$$\delta : A \rightarrow \ker \pi, a \mapsto ab \otimes 1 - b \otimes a$$

then it is easy to see that δ is a *b-derivation*. Indeed $\delta = b\delta'$ where

$$\delta' : A \rightarrow \ker \pi, a \mapsto a \otimes 1 - 1 \otimes a$$

since A is *b-contractible*, then there exists an element $\sum_n c_n \otimes d_n \in \ker \pi$ such that

$$\delta(a) = \sum_n ac_n \otimes d_n - \sum_n c_n \otimes d_n a \quad a \in A.$$

Let $d = b \otimes 1 - \sum_n c_n \otimes d_n$. The above identities show that $\pi(d) = b$ and $a.d = d.a$ for all $a \in A$. Therefore, d is a b -diagonal for A .
 Conversely suppose $d = \sum_n a_n \otimes b_n$ is a b -diagonal for A , X is an A -module and $\delta : A \rightarrow X$ is a bounded derivation. Clearly the map

$$\psi : A \times A \rightarrow X, (a, c) \mapsto a\delta(c)$$

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map $\varphi : A \hat{\otimes} A \rightarrow X$ such that $\varphi \circ \otimes = \psi$ that is $\varphi(a \otimes c) = a\delta(c)$. In particular using the fact that d is a b -diagonal for A , we get

$$\sum_n aa_n\delta(b_n) = \varphi(a.d) = \varphi(d.a) = \sum_n a_n\delta(b_na), \quad a \in A.$$

Now if $x = \sum_n a_n\delta(b_n)$, then for every $a \in A$ we have

$$ax - xa = \sum_n aa_n\delta(b_n) - \sum_n a_n\delta(b_n)a = \sum_n aa_n\delta(b_n) + b\delta(a) - \sum_n a_n\delta(b_na).$$

Thus the identity $ax - xa = b\delta(a)$ holds for every $a \in A$. Therefore every b -derivation is inner.

Example 3.2. Let A be the Banach algebra $l_1(N)$ with pointwise multiplication and $\{e_n\}$ be the standard basis for A . Then for every positive integer n , A is e_n -contractible. Indeed $e_n \otimes e_n$ is an e_n -diagonal for A . But A is not contractible, since it is not unital. Therefore b -contractibility does not imply contractibility.

Remark. If A is contractible, then it is unital and one can easily observe that A is b -contractible for every $b \in A - \{0\}$. However the above example shows that for non-unital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

Problem. Does there exist a unital Banach algebra which is b -contractible for some nonzero central idempotent b , but is not contractible?

References

1. P. C. Curtis, Jr and R. J. Loy, The structure of amenable Banach algebras, J. London Math. Soc. 40 (1989) 89-104.

2. G. H. Esslamzadeh Banach, algebra structure and amenability of a class of matrix algebras with applications, *J. Funct. Anal.* 161 (1999) 364–383 .
3. G. H. Esslamzadeh and T. Esslamzadeh, contractibility of ℓ^1 -Munn algebras with applications, *Semigroup Forum* 63 (1) (2001) 1–10.
4. J. E. Gale, T. J. Ransford and M. C. White, Weakly compact homomorphisms, *Trans. Amer. Math. Soc.* 331 (1992) 815–824.
5. N. Gronbaek, Various notions of amenability, a survey of problems, *Proceedings of 13th International Conference on Banach Algebras in Blaubeuren, 1997*, Walter de Gruyter, Berlin (1998) 535–547.
6. B. E. Johnson, Cohomology in Banach algebras, *Mem. Amer. Math. Soc.* 127 (1972).
7. B. E. Johnson, Approximate diagonals and cohomology of certain annihilator Banach algebras, *Amer. J. Math.* 94 (1972) 685–698.
8. A. Y. Helemskii, Flat Banach modules and amenable algebras, *Trans. Moscow Math. Soc.* (1984), *Amer. Math. Soc. Translations* (1985) 199–244 .
9. A. Y. Helemskii, *The homology of Banach and topological algebras*, (Translated from Russian), Kluwer (1989)
10. A. J. Lazar, S. K. Tsui and S. Wright, A cohomological characterization of finite dimensional C^* -algebras, *J. Operator Theory* 14 (1985) 239–247.
11. T. W. Palmer, *The Banach algebras and the general theory of *-algebras*, Vol. I, algebras and Banach algebras, Cambridge Univ. Press(1994)
12. V. Runde, The structure of contractible and amenable Banach algebras, *Proceedings of 13th International Conference on Banach Algebras in Blaubeuren, 1997*, Walter de Gruyter, Berlin (1998) 415-430.
13. J. L. Taylor, Homology and cohomology for topological algebras, *Adv. in Math.* 9 (1972) 137–182.
14. Y. Zhang, Maximal ideals and the structure of contractible and amenable Banach algebras, *Bull. Aust. Math. Soc.* (2) 62 (2000) 221–226.