

# The Existence of a Topolinear Isomorphism on an infinite dimensional Hilbert Space $H$ Corresponding to a Homeomorphism on it's Projective Space $P(H)$

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## Abstract

In this paper we prove a theorem which states the relationship between the topolinear isomorphisms on an infinite dimensional Hilbert Space  $H$  and the Homeomorphisms on projective Space  $P(H)$ . This theorem is proved by E.Artin in the finite dimensional case.

**Key words:** Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.

# Introduction

The following  $H$  is an infinite dimensional separable Hilbert Space and  $P(H)$  is its projective space which is given a smooth structure as in [2]. We mean by  $[x] \in P(H)$  the one dimensional vector subspace of  $H$  generated by  $x \in \hat{H} = H - 0$ .

$[x] + [y]$  means the two dimensional subspace generated by  $x, y \in \hat{H}$ . in fact  $[z] \subset [x] + [y]$  means that there exists  $a, b \in \hat{R}$  such that  $z = ax + by$ . and if  $[z] \neq [x]$ , There exists a unique  $d \in \hat{R}$  such that  $[z] = [x + dy]$ . We quote some necessary statments from [2].

**Proposition 1.1** *Let  $S$  be a unit sphere in a normed linear space  $B$  and  $T : B \rightarrow B$  a linear bijection, and  $\tilde{T}$  be the induced bijection*

$$\tilde{T} : S \rightarrow S$$

defined by  $\tilde{T}(u) = \frac{T(u)}{\|T(u)\|}$  for  $u \in S \subset B$ . If  $\tilde{T}$  is a homeomorphism then  $T$  is also homeomorphism.

we are ready to state the theorem which is the main topic of this paper

**Proposition 1.2** *Let  $f : P(H) \rightarrow P(H)$  be a topological isomorphism such that*

$$[x] \subset [y] + [z] \rightarrow f[x] \subset f[y] + f[z].$$

Then there exists a topological isomorphism  $T : P(H) \rightarrow P(H)$  such that the induced transformation  $\tilde{T} : P(H) \rightarrow P(H)$  agrees with  $f$ .

**Proof.** the hypothesis implies that if  $[x] \subset [y] + [z]$  then  $f^{-1}[x] \subset f^{-1}[y] + f^{-1}[z]$  and by induction on  $k$ , we get that if  $[z] \subset [z_1] + \dots + [z_k]$  then  $f[z] \subset f[z_1] + \dots + [z_k]$ , and  $f^{-1}[z] \subset f^{-1}[z_1] + \dots + f^{-1}[z_k]$ .

Let  $\{x_i\}$  be a Hamel basis for  $H$  where  $i$  is an arbitrary element of a set  $\mathcal{A}$ . It is clear that if  $f[x_i] = [y_i]$  then  $\{y_i\}$  is also a Hamel basis for  $H$ .

Now we choose an element of  $\mathcal{A}$  call it 1, then for any  $i \neq 1$  the line

$$L_i = [x_1 + x_i] \subset [x_1] + [x_i]$$

where  $L_i$  is not coincide with  $[x_i]$  or  $[x_1]$ , consequently

$$fL_i \subset [y_1] + [y_i]$$

and  $fL_i$  is not coincide with  $[y_i]$  or  $[y_1]$ . Then, for some unique  $d_i \in \mathbb{R}$  we have

$$fL_i = [y_1 + d_i y_i].$$

by choosing a suitable  $y_i$  we may assume that  $d_i = 1$ . Then

$$\text{for } i \in \mathcal{A}, \quad f[x_i] = [y_i] \quad (1)$$

$$\text{and for } i \neq 1, \quad f[x_1 + x_i] = [y_1 + y_i].$$

Now we choose another index from  $\mathcal{A}$ , call it 2. Then for  $a \in \mathbb{R}$

$$L = [x_1 + ax_2] \subset [x_1] + [x_2] \text{ where } L \neq [x_2]$$

Therefore

$$fL \subset [y] + [y_2], \text{ where } fL \neq [y_2].$$

Then for a unique  $a' \in \mathbb{R}$  we have

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**Theorem 1.2** *Let  $f : P(H) \rightarrow P(H)$  be a topological isomorphism such that*

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Therefore

$$fL \subset [y] + [y_2], \text{ where } fL \neq [y_2].$$

Then for a unique  $a' \in \mathbb{R}$  we have

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Now we define

$$\mu : R \longrightarrow R$$

by  $\mu(a) = a'$  and we will show that  $\mu$  is the identity function on  $R$ . Since

$$[x_1 + ax_2] \neq [x_1 + bx_2] \text{ if } a \neq b$$

it follows that  $a' \neq b'$ , then  $\mu$  is injective. We have also from (1) that

$$0' = 0 \text{ and } 1' = 1. \quad (2)$$

Now, we will show that for any  $i \in \mathcal{A}$

$$f[x_1 + ax_i] = [y_1 + a'y_i]$$

For any fixed  $i \neq 1, 2$  in  $\mathcal{A}$  we have

$$f[x_1 + ax_i] = [y_1 + by_i].$$

On the other hand  $L = [ax_2 - ax_i] \subset [x_2] + [x_i]$  with  $L \neq [x_i]$ , and so  $fL \subset [y_2] + [y_i]$  with  $fL \neq [y_i]$ . Consequently,  $fL = [y_2 + dy_i]$  for some unique  $d$ . On the other hand,

$$L \subset [x_1 + ax_2] + [x_1 + ax_i] \text{ with } L \neq [x_1 + ax_i].$$

Then as before  $fL = ([y_1 + a'y_2] + d'(y_1 + by_i))$  and it follows that  $d = -\frac{b}{a'}$ . But

$$L \subset [x_1 + x_2] + [x_1 + x_i] \text{ with } L \neq [x_1 + x_i]$$

and by (1)

$$fL \subset [y_1 + y_2] + [y_1 + y_i] \text{ with } fL \neq [y_1 + y_i]$$

Then for some unique  $h$  we have  $fL = [y_1 + y_2 + h(y_1 + y_i)]$ , consequently  $d = -1$  and  $b = a'$ , then for all  $i \in \mathcal{A}$  and  $a \in R$  we have

$$f[x_1 + ax_i] = [y_1 + a'y_i]. \quad (3)$$

Now we are going to prove that  $\mu$  is surjective. Choose a finite number of  $n$  vectors of  $\mathcal{A}$  including  $x_1$  and  $x_2$  say  $x_1, x_2, \dots, x_n$ . Then by induction we have

$$f[x_1 + a_2x_2 + \dots + a_nx_n] = [y_1 + a'_2y_2 + \dots + a'_ny_n]$$

and it follows that

$$f[a_2x_2 + \dots + a_nx_n] = [a'_2y_2 + \dots + a'_ny_n].$$

[1] page 90.

Let  $L = [y_1 + by_2]$  be a point of  $P(H)$ , since  $f$  is bijective, then there exists some  $v \in \dot{H}$  such that  $L = f[v]$ , then  $v$  can be written as a linear combination of  $x_j$  including  $x_1, x_2$ . For this purpose we can use the above set  $x_1, x_2, \dots, x_n$  then

$$v = \alpha_1x_1 + \alpha_2x_2 + \dots + \alpha_nx_n.$$

By (5) we have  $\alpha_1 \neq 0$  and consequently,

$$L = f[x_1 + \beta_2x_2 + \dots + \beta_nx_n] \text{ with } \beta_j = \frac{\alpha_j}{\alpha_1}$$

Then by (4)  $\beta'_2 = b$  and consequently  $\mu$  is surjective.

To show that  $\mu(a + b) = \mu(a) + \mu(b)$  we consider the line  $L = [x_1 + (a + b)x_2 + x_3]$ . Then by (2) and (3) we have

$$fL = [y_1 + (a + b)'y_2 + y_3]$$

but

$$L \subset [x_1 + ax_2] + [bx_2 + x_3] \text{ and } L \neq [bx_2 + x_3].$$

By (4) and (5)

$$fL \subset [y_1 + a'y_2] + [by_2 + y_3]$$

and so  $fL = [(y_1 + a'y_2) + \lambda(b'y_2 + y_3)]$  for some  $\lambda$ . It follows that  $\lambda = 1$  and so

$$\mu(a + b) = (a + b)' = a' + b' = \mu(a) + \mu(b). \quad (6)$$

Similarly by considering a line  $[x_1 + (ab)x_2 + x_3]$ , we get

$$\mu(ab) = \mu(a) \cdot \mu(b) \quad (7)$$

Thus  $\mu$  is a bijective mapping satisfying (2), (6) and (7) and therefore it is the identity mapping on  $R$ . Consequently

$$f[a_1x_1 + \dots + a_kx_k] = [a_1y_1 + \dots + a_ky_k]. \quad (8)$$

The equation (8) has been derived by fixing  $x_1, x_2$  from the Hamel basis  $\{x_i\}$ . Since it still holds for  $a_1, a_2$  zeros, it follows that (8) is true for any finite combination of vectors in  $\{x_i\}$ .

If  $x \in H$ , then  $x = \sum a_i x_i$  (a finite sum) and so we define a linear map

$$T: H \longrightarrow H \text{ by } T(x) = \sum a_i y_i$$

then  $T$  is also a bijection and it induces a map

$$\bar{T}: P(H) \longrightarrow P(H)$$

$$\bar{T}[x] = [T(x)] = [\sum a_i y_i] = f[x]$$

Consequently,  $\bar{T}$  agrees with  $f$ .

Let the bijection  $\tilde{T}: S \longrightarrow S$  defined by  $T$  as in Theorem 1.1 is a homeomorphism. This follows from the commutative diagram

$$\begin{array}{ccc} P(H) & \xrightarrow{f} & P(H) \\ \phi \uparrow & & \uparrow \phi \\ S & \xrightarrow{\tilde{T}} & S \end{array} \quad (9)$$

because  $f$  is supposed a homeomorphism and  $\phi$  is the local diffeomorphism between  $S$  and  $P(H)$ , it follows from Theorem 1.1 that  $\tilde{T}$  is a

## References

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