The Existence of a Topolinear Isomorphism on an infinite dimensional Hilbert Space H Corresponding t a Homeomorphism on it's Projective Space $P(H)$

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Abstract

In this paper we prove a theorem which states the relationship between the topolinear isomorphisms on an infinite dimentional Hilbert Space H and the Homeomorphisms on projective Space $P(H)$. This theorem is proved by E.Artin in the finite dimentional case.

Key words: Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.

Introduction

e following H is an infinite dimentional able Hilbert Space and $P(H)$ is its pro- e space which is given a smooth struc- as in [2]. We mean by $[x] \in P(H)$ the nentional vector subspace of H generated $\in \hat{H} = H - 0$.

$]+[y]$ means the two dimentional subspace rated by $x, y \in \hat{H}$. in fact $[z] \subset [x] + [y]$ s that there exists $a, b \in \hat{R}$ such that $z =$ by. and if $[z] \neq [x]$, There exists a unique $\frac{b}{a}$ such that $[z] = [x + dy]$. We quote some ssary statmants from [2].

orem 1.1 *Let S be a unit sphere in a normed r space B and $T : B \longrightarrow B$ a linear bijec- ranformation, and $\tilde{T}$ be the induced bijec- ransformation*

$$\tilde{T} : S \longrightarrow S$$

ed by $\tilde{T}(u) = \frac{T(u)}{\|T(u)\|}$ for $u \in S \subset B$. is a homeomorphism then T is also home- rphism.

we are ready to state the theorem which is oal of this paper

orem 1.2 *Let $f : P(H) \longrightarrow P(H)$ be a eomorphism such that*

$$[x] \subset [y] + [z] \longrightarrow f[x] \subset f[y] + f[z].$$

there exists a topolinear isomorphism T : $\to H$ such that the induced transformation $P(H) \longrightarrow P(H)$ agrees with f.

Proof. the hypothesis implies that if $[x] \subset [y]+[z]$ then $f^{-1}[x] \subset f^{-1}[y]+f^{-1}[z]$ and by induction on k, we get that if $[z] \subset [z_1] + \cdots + [z_k]$ then $f[z] \subset f[z_1] + \cdots + [z_k]$, and $f^{-1}[z] \subset f^{-1}[z_1] + \cdots + f^{-1}[z_k]$.

Let $\{x_i\}$ be a Hamel basis for H where i is an arbitrary element of a set $\mathcal{A}$. It is clear that if $f[x_i] = [y_i]$ then $\{y_i\}$ is also a Hamel basis for H.

Now we choose an element of $\mathcal{A}$ call it 1,then for any $i \neq 1$ the line

$$L_i = [x_1 + x_i] \subset [x_1] + [x_i]$$

where L_i is not coinside with $[x_i]$ or $[x_1]$, consequently

$$fL_i \subset [y_1] + [y_i]$$

and fL_i is not coinside with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in$ R we have

$$fL_i = [y_1 + d_i y_i].$$

by choosing a suitable y_i we may assume that $d_i = 1$. Then

$$\text{for } i \in \mathcal{A}, \quad f[x_i] = [y_i] \qquad (1)$$
$$\text{and for } i \neq 1, \quad f[x_1 + x_i] = [y_1 + y_i].$$

Now we choose another index from $\mathcal{A}$, call it 2. Then for $a \in$ R

$$L = [x_1 + ax_2] \subset [x_1] + [x_2] \text{ where } L \neq [x_2]$$

Therefore

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Therefore

$$fL \subset [y] + [y_2], \quad \text{where} \quad fL \neq [y_2].$$

Then for a unique $a' \in R$ we have

Now we define

$$\mu : R \longrightarrow R$$

by $\mu(a) = a'$ and we will show that μ is the identity function on R. Since

$$[x_1 + ax_2] \neq [x_1 + bx_2] \quad \text{if} \quad a \neq b$$

it follows that $a' \neq b'$, then μ is injective. We have also from (1) that

$$0' = 0 \quad \text{and} \quad 1' = 1. \qquad (2)$$

Now, we will show that for any $i \in \mathcal{A}$

$$f[x_1 + ax_i] = [y_1 + a'y_i]$$

For any fixed $i \neq 1, 2$ in $\mathcal{A}$ we have

$$f[x_1 + ax_i] = [y_1 + by_i].$$

On the other hand $L = [ax_2 - ax_i] \subset [x_2] + [x_i]$ with $L \neq [x_i]$, and so $fL \subset [y_2] + [y_i]$ with $fL \neq [y_i]$. Consequently, $fL = [y_2 + dy_i]$ for some unique d. On the other hand,

$$L \subset [x_1 + ax_2] + [x_1 + ax_i] \quad \text{with} \quad L \neq [x_1 + ax_i].$$

Then as before $fL = ([y_1 + a'y_2) + d'(y_1 + by_i)]$ and it follows that $d = -\frac{b}{a'}$. But

$$L \subset [x_1 + x_2] + [x_1 + x_i] \quad \text{with} \quad L \neq [x_1 + x_i]$$

and by (1)

$$fL \subset [y_1 + y_2] + [y_1 + y_i] \quad \text{with} \quad fL \neq [y_1 + y_i]$$

Then for some unique h we have $fL = [y_1 + y_2 + h(y_1 + y_i)]$, consequently $d = -1$ and $b = a'$, then for all $i \in \mathcal{A}$ and $a \in R$ we have

$$f[x_1 + ax_i] = [y_1 + a'y_i]. \qquad (3)$$

Now we are going to prove that μ is surj-tive. Choose a finite number of n vectors of $\{$ including x_1 and x_2 say $x_1, x_2, \cdots, x_n$. Then induction we have

$$f[x_1 + a_2x_2 + \cdots + a_nx_n] = [y_1 + a_2'y_2 + \cdots + a_n'$$

and it follows that

$$f[a_2x_2 + \cdots + a_nx_n] = [a_2'y_2 + \cdots + a_n'y_n]. \quad ($$

[1] page 90.

Let $L = [y_1 + by_2]$ be a point of $P(H)$, since is bijective, then there exists some $v \in \hat{H}$ suc that $L = f[v]$, then v can be written as a lin ear combination of x_j including x_1, x_2. For th purpose we can use the above set $x_1, x_2, \cdots, x$ then

$$v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.$$

By (5) we have $\alpha_1 \neq 0$ and consequently,

$$L = f[x_1 + \beta_2 x_2 + \cdots + \beta_n x_n] \quad \text{with} \quad \beta_j = \frac{\alpha_j}{\alpha_1}$$

Then by (4) $\beta_2' = b$ and consequently μ is sur-jective.

To show that $\mu(a + b) = \mu(a) + \mu(b)$ we consider the line $L = [x_1 + (a + b)x_2 + x_3]$. Then by (2) and (3) we have

$$fL = [y_1 + (a + b)'y_2 + y_3]$$

but

$$L \subset [x_1 + ax_2] + [bx_2 + x_3] \quad \text{and} \quad L \neq [bx_2 + x_3].$$

By (4) and (5)

$$fL \subset [y_1 + a'y_2] + [b' \ldots$$

nd so $fL = [(y_1 + a'y_2) + \lambda(b'y_2 + y_3)]$ for some . It follows that $\lambda = 1$ and so

$$\mu(a+b) = (a+b)' = a' + b' = \mu(a) + \mu(b). \quad (6)$$

Similarly by considering a line $[x_1 + (ab)x_2 +$ $x_3]$, we get

$$\mu(ab) = \mu(a).\mu(b) \quad (7)$$

Thus μ is a bijective mapping satisfying (2),(6) nd (7) and therefore it is the identity mapping R. Consequently

$$f[a_1x_1 + \cdots + a_kx_k] = [a_1y_1 + \cdots + a_ky_k]. \quad (8)$$

The equation (8) has been derived by fixing $_1, x_2$ from the Hamel basis $\{x_i\}$. Since it still olds for a_1, a_2 zeros, It follows that (8) is true r any finite combination of vectors in $\{x_i\}$.

If $x \in H$, then $x = \sum a_ix_i$ (a finite sum) nd so we define a linear map

$$T : H \longrightarrow H \quad \text{by} \quad T(x) = \sum a_iy_i$$

hen T is also a bijection and it induces a map

$$\overline{T} : P(H) \longrightarrow P(H)$$

$$\overline{T}[x] = [T(x)] = [\sum a_iy_i] = f[x]$$

onsequently, $\overline{T}$ agrees with f.

he bijection $\tilde{T} : S \longrightarrow S$ defined by T as in heorem 1.1 is a homeomorphism. This follows om the commutative diagram

$$\begin{array}{ccc} P(H) & \xrightarrow{f} & P(H) \\ \phi \uparrow & & \uparrow \phi \\ S & \xrightarrow{\tilde{T}} & S \end{array} \quad (9)$$

ecause f is supposed a homeomorphism and is the local diffeomorphism between S and (H), it follows from Theorem 1.1 that T is a

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