

The Existence of a Topolinear Isomorphism on an infinite dimensional Hilbert Space H Corresponding a Homeomorphism on it's Projective Space P(H)

Ebrahim Esrafilian

Department of Mathematics

Iran University of Science and Technology

Narmak, Tehran-16, Iran

Abstract

In this paper we prove a theorem which states the relationship between the topolinear isomorphisms on an infinite dimentional Hilbert Space H and the Homeomorphisms on projective Space P(H). This theorem is proved by E.Artin in the finite dimentional case.

Key words: Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.

Table 3 Patented from John who

Introduction

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[z]+[y] means the two dimentional subspace rated by $x,y \in \hat{H}$. in fact $[z] \subset [x]+[y]$ is that there exists $a,b \in \hat{R}$ such that z=by and if $[z] \neq [x]$, There exists a unique [z] such that [z]=[x+dy]. We quote some sary statements from [2].

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we are ready to state the theorem which is

proper 1.2 Let $f: P(H) \longrightarrow P(H)$ be a conformal such that

$$[x] \subset [y] + [z] \longrightarrow f[x] \subset f[y] + f[z].$$

Refere exists a topolinear isomorphism $T: \rightarrow H$ such that the induced transformation $P(H) \longrightarrow P(H)$ agrees with f.

Proof. the hypothesis implies that if $[x] \subset [y]+[z]$ then $f^{-1}[x] \subset f^{-1}[y]+f^{-1}[z]$ and by induction on k, we get that if $[z] \subset [z_1]+\cdots+[z_k]$ then $f[z] \subset f[z_1]+\cdots+[z_k]$, and $f^{-1}[z] \subset f^{-1}[z_1]+\cdots+f^{-1}[z_k]$.

Let $\{x_i\}$ be a Hamel basis for H where i is an arbitrary element of a set A. It is clear that if $f[x_i] = [y_i]$ then $\{y_i\}$ is also a Hamel basis for H.

Now we choose an element of A call it 1,then for any $i \neq 1$ the line

$$L_i = [x_1 + x_i] \subset [x_1] + [x_i]$$

where L_i is not coinside with $[x_i]$ or $[x_1]$, consequently

$$fL_i \subset [y_1] + [y_i]$$

and fL_i is not coinside with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in \mathbb{R}$ we have

$$fL_i = [y_1 + d_i y_i].$$

by choosing a suitable y_i we may assume that $d_i = 1$. Then

for
$$i \in A$$
, $f[x_i] = [y_i]$ (1)
and for $i \neq 1$, $f[x_1 + x_i] = [y_1 + y_i]$.

Now we choose another index from A, call it 2. Then for $a \in \mathbb{R}$

$$L = [x_1 + ax_2] \subset [x_1] + [x_2]$$
 where $L \neq [x_2]$

Therefore

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Then for a unique $a' \in R$ we have

Now we define

$$\mu : R \longrightarrow R$$

by $\mu(a) = a'$ and we will show that μ is the identity function on R. Since

$$[x_1 + ax_2] \neq [x_1 + bx_2]$$
 if $a \neq b$

it follows that $a' \neq b'$, then μ is injective. We have also from (1) that

$$0' = 0$$
 and $1' = 1$. (2) [1] page 90.

Now, we will show that for any $i \in \mathcal{A}$

$$f[x_1 + ax_i] = [y_1 + a'y_i]$$

For any fixed $i \neq 1, 2$ in \mathcal{A} we have

$$f[x_1 + ax_i] = [y_1 + by_i].$$

On the other hand $L = [ax_2 - ax_i] \subset [x_2] + [x_i]$ with $L \neq [x_i]$, and so $fL \subset [y_2] + [y_i]$ with $fL \neq [y_i]$. Consequently, $fL = [y_2 + dy_i]$ for some unique d. On the other hand,

$$L \subset [x_1 + ax_2] + [x_1 + ax_i]$$
 with $L \neq [x_1 + ax_i]$.

Then as before $fL = ([y_1 + a'y_2) + d'(y_1 + by_i)]$ and it follows that $d = -\frac{b}{a'}$. But

$$L \subset [x_1 + x_2] + [x_1 + x_i]$$
 with $L \neq [x_1 + x_i]$

and by (1)

$$fL \subset [y_1 + y_2] + [y_1 + y_i]$$
 with $fL \neq [y_1 + y_i]$

Then for some unique h we have $fL = [y_1 + y_2 + h(y_1 + y_i)]$, consequently d = -1 and b = a', then for all $i \in A$ and $a \in R$ we have

 $f[x_1 + ax_i] = [y_1 + a'y_i].$

Now we are going to prove that
$$\mu$$
 is surjetive. Choose a finite number of n vectors of $\{x_1, x_2, \dots, x_n\}$. Then induction we have

$$f[x_1 + a_2x_2 + \dots + a_nx_n] = [y_1 + a_2'y_2 + \dots + a_n']$$

and it follows that

$$f[a_2x_2 + \dots + a_nx_n] = [a'_2y_2 + \dots + a'_ny_n].$$
 (

Let $L = [y_1 + by_2]$ be a point of P(H), since is bijective, then there exists some $v \in \hat{H}$ such that L = f[v], then v can be written as a linear combination of x_j including x_1, x_2 . For the purpose we can use the above set x_1, x_2, \dots, x_n

$$v = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n.$$

By (5) we have $\alpha_1 \neq 0$ and consequently,

$$L = f[x_1 + \beta_2 x_2 + \dots + \beta_n x_n] \text{ with } \beta_j = \frac{\alpha_j}{\alpha_1}.$$

Then by (4) $\beta_2' = b$ and consequently μ is surjective.

To show that $\mu(a + b) = \mu(a) + \mu(b)$ we consider the line $L = [x_1 + (a+b)x_2 + x_3]$. Then by (2) and (3) we have

$$fL = [y_1 + (a+b)'y_2 + y_3]$$

but

$$L \subset [x_1 + ax_2] + [bx_2 + x_3]$$
 and $L \neq [bx_2 + x_3]$.

By (4) and (5)

(3)

and so $fL = [(y_1 + a'y_2) + \lambda(b'y_2 + y_3)]$ for some It follows that $\lambda = 1$ and so

$$u(a+b) = (a+b)' = a' + b' = \mu(a) + \mu(b).$$
 (6)

Similarly by considering a line $[x_1+(ab)x_2+$ r_3], we get

$$\mu(ab) = \mu(a).\mu(b)$$
 (7)

Thus μ is a bijective mapping satisfying (2),(6) nd (7) and therefore it is the identity mapping R. Consequently

$$f[a_1x_1 + \dots + a_kx_k] = [a_1y_1 + \dots + a_ky_k].$$
 (8)

The equation (8) has been derived by fixing x_1, x_2 from the Hamel basis $\{x_i\}$. Since it still olds for a_1, a_2 zeros, It follows that (8) is true r any finite combination of vectors in $\{x_i\}$.

If $x \in H$, then $x = \sum a_i x_i$ (a finite sum) nd so we define a linear map of the ballous state

where
$$T:H\longrightarrow H$$
 by $T(x)=\sum_i a_iy_i$ is a manifestanting results find because media

hen T is also a bijection and it induces a map

$$\overline{T}: P(H) \longrightarrow P(H)$$

$$\overline{T}[x] = [T(x)] = [\sum a_i y_i] = f[x]$$

 $T: P(H) \longrightarrow P(H)$ 7.202 $\overline{T}[x] = [T(x)] = [\sum a_i y_i] = f[x]$ 8.224 on sequently, \overline{T} agrees with f.
8.255 The bijection $\widetilde{T}: S \longrightarrow S$ defined by T as in herem 1.1 is a homeomorphism. This follows

the commutative diagram
$$P(H) \xrightarrow{f} P(H) \qquad (9)$$

$$\phi \uparrow \qquad \uparrow \phi$$

$$S \xrightarrow{\hat{T}} S$$

ecause f is supposed a homeomorphism and is the local diffeomorphism between S and VIII to full many from Philosophy 1 1 Albert III to

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