ORTHOGONALIZATION PROCESS FOR FINDING A BASIC FEASIBLE SOLUTION (BFS)

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Abstract

For finding an optimal solution in L.P., combination of orthogonality and simplex method is used. It seems that the number of iteration is reduced with respect to new algorithm in [1].

1. INTRODUCTION.

The simplex method for solving L.P. was developed by G.B. Dantzig. This is an iterative method which comes to the solution or shows that problem has no feasible solution. Here we consider linear programming in canonical form (for example in $\mathcal{R}^2$):

$$\begin{aligned} Maximize \quad z = \quad & c_1x_1 + c_2x_2, \\ s.t. \quad & AX \leq B, \\ & X \geq 0, \\ & X = (x_1, x_2). \end{aligned} \tag{1}$$

For simplicity it has been assumed that $\vec{C} = (c_1, c_2) \geq 0$. The vector $\lambda\vec{C}(\lambda \in \mathcal{R})$ is perpendicular to the hyperplane $c_1x_1 + c_2x_2 = z_0$. Suppose that the vector $\lambda\vec{C}$ intersects the feasible region boundary at points F_1 and F_2 (Figure 1).

If we start from point F_2, at most two iterations, are needed for finding the optimal solution. This is the main idea which will be used in this paper, i.e., finding an optimal solution with combination of orthogonality and simplex method. The theory will be discussed in section 2. In section 3 some examples will be presented.

2. ALGORITHM.

Consider

$$Maximize\ z = \tag{2}$$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

Subject to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i,$$

$$i = p_1 + 1, \ldots, p_2$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i,$$

$$i = p_2 + 1, \ldots, m$$

$$x_j \geq 0 \ ; \ j = 1, \ldots, n.$$

It has been assumed that $b_i > 0$ for all i. After adding slacks and surpluses we have

$$Maximize\ z = \tag{3}$$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

Subject to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + x_{n+i} = b_i,$$

$$i = 1, \ldots, p_1$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - x_{n+i} = b_i,$$

$$i = p_1 + 1, \ldots, p_2$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad = b_i,$$

$$i = p_2 + 1, \ldots, m$$

$$x_j \geq 0 \ ; \ j = 1, \ldots, n + p_2.$$

There are two cases.

Case 1:

$p_2 = m$; that means there are no equality constraints, we define:

$$P = \{ j \mid c_j > 0 \},$$

$$N = \{ 1, 2, \ldots, n \},$$

$$\alpha_i = \sum_{j=1}^{n} c_j a_{ij} \ ; \ i = 1, 2, \ldots, m,$$

$$Q = \{ i \mid \alpha_i \neq 0 \},$$

b_i

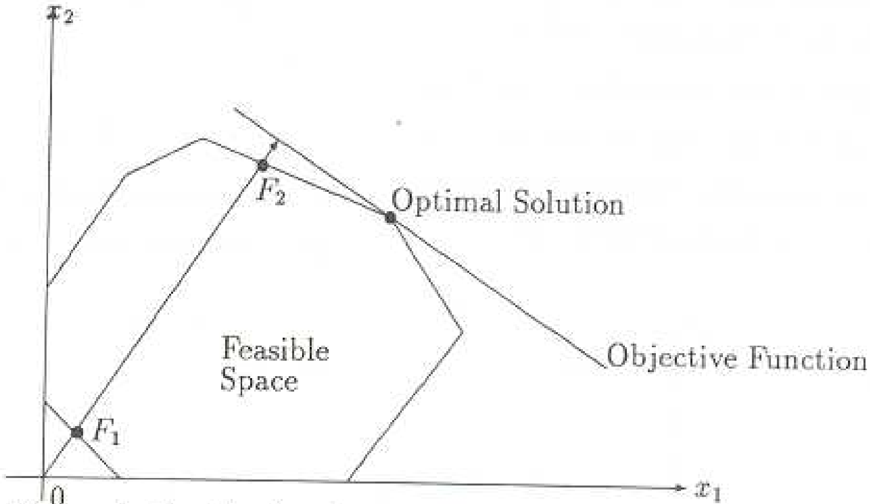

Figure 1. Combination of orthogonality and simplex method.

If P = N,

$$t_1^* = \min\ \{t_i > 0 \mid i \in Q \cap \{1, 2, \ldots, p_1\}\}$$
$$t_2^* = \max\ \{t_i > 0 \mid i \in Q \cap \{p_1 + 1, \ldots, m\}\}$$

If P $\neq$ N,

$$t_1^* = \min\ \{t_i \mid i \in Q \cap \{1, 2, \ldots, p_1\}\}$$
$$t_2^* = \max\ \{t_i \mid i \in Q \cap \{p_1 + 1, \ldots, m\}\}.$$

Here, α_i is inner product of objective function and i^{th} constraint. It can be proved $t_i = \|\vec{v}\|_2/\|\vec{u}\|_2$, where $\vec{v} = \lambda_1\vec{C}$ intersects i^{th} constraint and $\vec{u} = \lambda_2\vec{C}$ is projection of normal of i^{th} constraint on $\vec{C}$.

If $t_1^* \geq t_2^* \geq 0$, two feasible solutions are obtained as follows.

$$X^1 = (x_1^1, \ldots, x_n^1, b_1 - \alpha_1 t_1^*, \ldots,$$
$$b_{p_1} - \alpha_{p_1} t_1^*, \alpha_{p_1+1} t_1^* - b_{p_1+1},$$
$$\ldots, \alpha_m t_1^* - b_m)^t,$$

$$X^2 = (x_1^2, \ldots, x_n^2, b_1 - \alpha_1 t_2^*, \ldots,$$
$$\ldots, \alpha_m t_2^* - b_m)^t,$$

$$x_j^1 = c_j t_1^* \ \ , \ \ x_j^2 = c_j t_2^* \ \ , \ \ j = 1, \ldots, n.$$

In this case $\sum_{j=1}^n c_j x_j^1 \geq \sum_{j=1}^n c_j x_j^2$, so X^1 which gives better value for objective function, and we select it as initial feasible solution. If $t_2^* > t_1^*$, it can not be deduced that the constraints are inconsistent. In this case a solution is obtained, X^2, which is not feasible. See Figure 2.

However after doing orthogonality operation a feasible or infeasible solution which may be basic or non-basic is obtained, it is tried to get a BFS (Basic Feasible Solution) from this solution (if problem is unbounded or feasible).

Case 2:

$p_2 < m$; that means there are some equality constraints. Notice that each equality

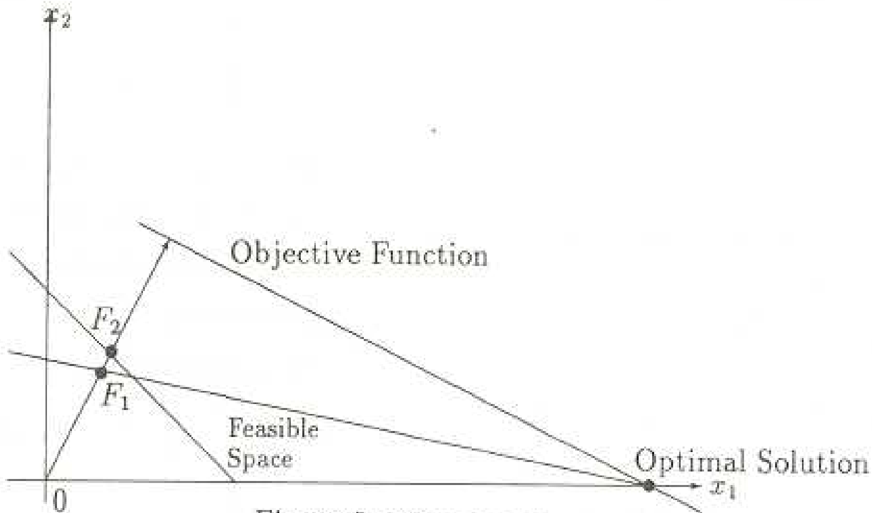

Figure 2. When $t_2^* > t_1^*$.

Case 1. A method for extracting a basic and feasible solution will be discussed as follows.

Basic Operation:

Consider (3) as follows:

$$\begin{aligned} Maximize \quad z = \quad & CX, \qquad (4)\\ s.t. \quad & AX = b,\\ & X \geq 0, \end{aligned}$$

where X contains all original, slack, surplus variables. Let X^1 be a solution vector obtained in orthogonality operation, i.e.,

$$AX^1 = b.$$

Suppose

$$\begin{aligned} x_j^1 &\neq 0 \quad j \in \{j_1, \ldots, j_r\},\\ &= 0 \quad Otherwise. \end{aligned}$$

then $\{A_{.j_1}, \ldots, A_{.j_r}\}$ is a set of column vectors of A, which are used by X^1. if this set is linearly independent, so X^1 is basic solution and feasibility is in our concern, otherwise zero, such that

$$\theta_1 A_{.j_1} + \theta_2 A_{.j_2} + \cdots + \theta_r A_{.j_r} = 0 \qquad (a)$$

as X^1 is a solution of (4), so

$$x_{j_1}^1 A_{.j_1} + x_{j_2}^1 A_{.j_2} + \cdots + x_{j_r}^1 A_{.j_r} = b \qquad (b)$$

from (a) and (b),

$$\begin{aligned} &(x_{j_1}^1 + \gamma\theta_1)A_{.j_1} + \qquad (5)\\ &(x_{j_2}^1 + \gamma\theta_2)A_{.j_2} + \cdots +\\ &(x_{j_r}^1 + \gamma\theta_r)A_{.j_r} = b, \end{aligned}$$

for all $\gamma \in \mathcal{R}$. Hence if

$$x_j^1(\gamma) = \begin{cases} x_{j_t}^1 + \gamma\theta_t & j = j_t, t = 1, \ldots, r\\ 0 & Otherwise \end{cases}$$

then $X^1(\gamma) = (x_1^1(\gamma), \ldots, x_{n+m}^1(\gamma))^t$ is a solution of (4) for all $\gamma \in \mathcal{R}$. By appropriate choice of γ, one of the columns of A which possibility for $X^1(\gamma)$ to be feasible, for this purpose, γ is chosen such that

$$x^1_{j_t} + \gamma\theta_t \geq 0 \qquad \forall\, t = 1, 2, \ldots, r$$

Suppose

$$\begin{aligned} \gamma_1 &= max\ \{-x^1_{j_t}/\theta_t : t \ such\ that \\ &\qquad \theta_t > 0\} \qquad (6) \\ &= -\infty \qquad if\ \theta_t \leq 0\ \forall\, t, \end{aligned}$$

$$\begin{aligned} \gamma_2 &= min\ \{-x^1_{j_t}/\theta_t : t \ such\ that \\ &\qquad \theta_t < 0\} \\ &= +\infty \qquad if\ \theta_t \geq 0\ \forall\, t. \end{aligned}$$

Now if $\gamma_1 \leq \gamma_2$, then for all value of γ satisfying $\gamma_1 \leq \gamma \leq \gamma_2$, $X^1(\gamma)$ is a feasible solution of (4). Since $(\theta_1, \theta_2, \ldots, \theta_r) \neq 0$ at least one of the value of γ_1 *or* γ_2 must be finite. Suppose γ is γ_1 *or* γ_2 which is finite, it is clear that in this case $X^1(\gamma)$ is a feasible solution, in which at most (r-1) value are different from zero. Now by repeating this procedure X^1 can be made basic solution. If in one of the iteration $\gamma_1 \leq \gamma_2$, then X^1 is made a basic feasible solution. When both γ_1 and γ_2 are finite, one of those should be chosen, which one is better? This choice is illustrated in two step:

A) If $\gamma_1 \leq \gamma_2$, then for all γ , $\gamma_1 \leq \gamma \leq \gamma_2$, $X^1(\gamma)$ is feasible, when the objective function is linear, one of the values γ_1 or γ_2 which give better value for objective function is chosen:

Maximization case,

$$\gamma = \begin{cases} \gamma_1 & if \quad CX^1(\gamma_1) \geq CX^1(\gamma_2) \\ \gamma_2 & if \quad CX^1(\gamma_1) < CX^1(\gamma_2) \end{cases}$$

Minimization case,

$$\gamma = \begin{cases} \gamma_1 & if \quad CX^1(\gamma_1) \leq CX^1(\gamma_2) \\ \gamma_2 & if \quad CX^1(\gamma_1) > CX^1(\gamma_2) \end{cases}$$

B) If $\gamma_1 > \gamma_2$, then for all γ , $\gamma_1 \geq \gamma \geq \gamma_2$, $X^1(\gamma)$ is infeasible solution, in this case the method for choosing γ is opposite what is done in step (a).

Maximization case,

$$\gamma = \begin{cases} \gamma_1 & if \quad CX^1(\gamma_1) \leq CX^1(\gamma_2) \\ \gamma_2 & if \quad CX^1(\gamma_1) > CX^1(\gamma_2) \end{cases}$$

Minimization case,

$$\gamma = \begin{cases} \gamma_1 & if \quad CX^1(\gamma_1) \geq CX^1(\gamma_2) \\ \gamma_2 & if \quad CX^1(\gamma_1) < CX^1(\gamma_2) \end{cases}$$

Of course this choice of γ depends to the nature of problem. The experience shows that the above mentioned method is rather successful.

On the whole after using orthogonal operation exactly in two iteration of Gaussian elimination a basic solution, which is denoted by X^1 here after, is obtained.

Feasibility Operation:

In the end of basic operation instead of BFS, a BS may by obtained, i.e., some of the components of X^1 may be negative. In this case by operation similar to dual simplex method a feasible solution is obtained.

3. EXAMPLES.

$$\begin{aligned} &\textit{Subject to} \\ &2x_1 + x_2 \le 10 \\ &x_1 + 2x_2 \le 10 \\ &4x_1 - 2x_2 \ge 1 \\ &-2x_1 + 4x_2 \ge 1 \\ &x_i \ge 0 \ , \ i = 1,2. \end{aligned}$$

2)

$$Maximize \ \ x_1 + 5x_2 + 3x_3$$

$$\begin{aligned} &\textit{Subject to} \\ &x_1 + 2x_2 + x_3 \le 3 \\ &2x_1 - x_2 = 4 \\ &x_i \ge 0 \ , \ i = 1,2,3. \end{aligned}$$

3)

$$Maximize \ \ 3x_1 + 5x_2 + x_3 + x_4$$

$$\begin{aligned} &\textit{Subject to} \\ &x_1 + 2x_2 + x_3 + x_4 \le 40 \\ &5x_1 + x_2 \le 12 \\ &x_3 + 5x_4 \le 50 \\ &x_3 + x_4 \ge 5 \\ &x_i \ge 0 \ , \ i = 1,2,3,4. \end{aligned}$$

4)

$$Maximize \ \ 6x_1 + 7x_2 + 3x_3 + 5x_4 + x_5 + x_6$$

$$\begin{aligned} &\textit{Subject to} \\ &x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 50 \\ &x_1 + x_2 \le 10 \\ &x_2 \le 8 \\ &5x_3 + x_4 \le 12 \\ &x_5 + 5x_6 \le 50 \\ &x_5 + x_6 \ge 5 \end{aligned}$$

5)

$$Maximize \ \ 5x_1 + 3x_2 + 8x_3 - 5x_4$$

$$\begin{aligned} &\textit{Subject to} \\ &x_1 + 2x_2 + x_3 + x_4 \ge 25 \\ &5x_1 + x_2 \le 20 \\ &5x_1 - x_2 \ge 5 \\ &x_3 + x_4 = 20 \\ &x_i \ge 0 \ , \ i = 1,2,3,4. \end{aligned}$$

6)

$$\begin{aligned} &Maximize && -x_1 + 2x_2 \\ &\textit{Subject to} && x_1 + 2x_2 \le 12 \\ & && x_1 - x_2 \ge 2 \\ & && x_i \ge 0 \ , \ i = 1,2. \end{aligned}$$

7)

$$\begin{aligned} &Minimize && 2x_1 + x_2 \\ &\textit{Subject to} && 3x_1 + x_2 = 3 \\ & && x_1 + 2x_2 \le 3 \\ & && 4x_1 + 3x_2 \ge 6 \\ & && x_i \ge 0 \ , \ i = 1,2. \end{aligned}$$

8)

$$\begin{aligned} &Minimize && 2x_1 + 3x_2 - 5x_3 \\ &\textit{Subject to} && x_1 + x_2 + x_3 = 7 \\ & && 2x_1 - 5x_2 + x_3 \ge 10 \\ & && x_i \ge 0 \ , \ i = 1,2,3. \end{aligned}$$

9)

$$\begin{aligned} &Minimize && 5x_1 - 6x_2 - 7x_3 \\ &\textit{Subject to} && x_1 + 5x_2 - 3x_3 \ge 15 \\ & && 5x_1 - 6x_2 + 10x_3 \le 12 \\ & && x_1 + x_2 + x_3 = 5 \\ & && x_i \ge 0 \ , \ i = 1,2,3. \end{aligned}$$

Table 1

	Orthogonal and Simplex		Others		
Problem No.	Iterations of Dual S.	Iterations of Simplex	Big M	Decomposition Method	LINDO
1	0	0	3	-	3
2	1	0	3	-	2
3	1	0	4	4	3
4	1	0	6	4	5
5	0	1	4	4	3
6	0	0	2	-	2
7	0	1	2	-	3
8	0	0	4	-	2
9	0	0	4	-	4

Number of iterations in several methods

Here we solve each problem by a special method related to the problem. Each problem is solved by LINDO software (Table 1).

Note 1:In table 1 each problem in orthogonal and simplex method uses one Gaussian elimination iterative for **Basic Operation**.

Note 2:In the third column of table 1, each number is 0 or 1, and this is very important.

4. CONCLUSIONS.

In all examples our method works well than others. In examples 1,6,8 and 9 without any other iterations, after finding a BFS, the optimal solution is obtained. In other examples after one iteration, the optimal solution is obtained.

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An Augmented Galerkin Algorithm for First Kind Integral Equations of Hammerstein Type

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Abstract

Recent papers, [1],[2] & [3], describe some algorithms for linear first kind integral equations. These algorithms are based on augmented Galerkin method and Cross-validation scheme [5]. The results show that, these algorithms work well for linear equations.

In this paper we apply algorithms of [1] & [2] on non-linear first kind integral equations of Hammerstein type with bounded solution. In order to obtain a posteriori error estimate, we apply fifteen-point Gauss-Kronrod quadrature rule [4]. Finally, we give a number of numerical examples showing that the algorithms work well in practice.

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