A Complete Efficiency Ranking of Decision Making Units in DEA: with an Empirical Study

G. R. JAHANSHAHLOO
and
S. MEHRABIAN

Institute of Mathematics,
University for Teacher Education.

Abstract

The efficiency measures provided by DEA can be used for ranking Decision Making Units (DMU's), but this ranking cannot be applied to efficient units. Anderson and Peterson have proposed a modified efficiency measure for efficient units which can be used for ranking, but this ranking breaks down in case of units with at least one input of zero. This paper proposes an alternative efficiency measure that removes this problem. The model is illustrated by an application to the University for Teacher Education, for which the Anderson - Peterson model was not able to give a ranking for two units, which were ranked successfully by the proposed model.

(Data Envelopment Analysis, Efficiency, Ranking)

ntroduction.

Envelopment Analysis (DEA) provides a measure of the efficiency of a Decision Making Unit relative to other such units producing the outputs with the same inputs. This technique, eloped by Cha-rnes, Cooper, and Rhodes (1978), and extended by Banker, Charnes, ooper (BCC) (1984), is a linear programming ure for an analysis of inputs and outputs. The ure does not require prior weights on inputs tputs.

standard DEA method assigns an efficiency ess than one to inefficient DMU's, from which ing can be derived. However, efficient DMU's e an efficiency of 1, so that for these units no g can be given. A model for ranking efficient s was proposed by Andersen and Petersen . Their model was called Extended-DEA, and d in this study for the University for Teacher tion (UTE). However, this model breaks down icient units with at least one zero input.

his paper, a new definition of efficiency is pro- that can be extended for ranking efficient 's. The extended method is applied to data for TE.

e role of zeros in data has been considered by nes, Cooper and Thrall (1991) and Thomp- Dharmaphala, and Thrall (1993) but this pa- eals with the problem of ranking the efficient 's involving zeros in input data.

e paper unfolds as follows. Section 2 represents ndersen and Petersen model. Section 3 presents nodel based on a definition of efficiency in pro- on possibility set (PPS). In section 4, the two els are compared, using two illustrative example. on 5 applies the two models to the UTE data. mmary is given in section 6.

2 The Andersen-Petersen Model.

The standard DEA method assigns an efficiency score of less than one to inefficient units. A score less than one means that a linear combination of the other units could produce at least the same vector of output using a smaller vector of inputs. This score can be used to rank inefficient units. Andersen and Petersen (1993) developed a similar model for ranking efficient DMU's, which in the standard DEA method have a score of 1. The basic idea in their model is to compare the unit under evaluation with a linear combination of all other units, i.e., all units excluding the unit itself. In this case, an efficiency score above 1 is obtained for efficient units. This score reflects the radial distance from the unit under evaluation to the production frontier estimated with the exclusion of that unit, i.e., the maximum proportional increase in inputs producing at least the same outputs.

The Andersen-Petersen model (AP-Model) is identical with the CCR method, except that the unit under evaluation is not included in the combination. Therefore the p^{th} DMU can be evaluated as follows:

$$
\begin{aligned}
& r_p^* = \min \ r_p - \epsilon \left[\textstyle\sum_{i=1}^{m} s_i \ + \ \sum_{r=1}^{s} s_r^{'}\right] \\
& \text{subject to:} \\
& \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j X_{ij} + s_i = r_p X_{ip}, \quad i = 1, \ldots, m, \\
& \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j Y_{rj} - s_r^{'} = Y_{rp}, \quad r = 1, \ldots, s, \\
& \lambda_j, \ s_i, \ s_r^{'} \geq 0, \quad \forall j, \ i, \ r,
\end{aligned}
$$

where Y_{rj} is the r^{th} output and X_{ij} is the i^{th} input for the j^{th} DMU, r_p is a scalar defining the share of p^{th} DMU input vector which is required in order to produce the output vector of p^{th} DMU, λ_j denotes the intensity of the j^{th} DMU, and ϵ is an non-Archimedian infinitesimal.

3 Efficiency Analysis by an Alternative Measure.

There are n DMU's to be evaluated, each consumes varying amounts of m different inputs to produce s different outputs.

In the model formulation, X_p and Y_p denote, respectively, the nonnegative vectors of input and output values for DMU_p.

Definition. The *production possibility set* (PPS) T is the set $\{(X_t, Y_t)|$ the outputs Y_t can be produced with the inputs $X_t\}$.

The set of n DMU's of actual production possibility (X_j, Y_j), $j = 1, \ldots, n$ is considered. Our focus is on the empirically defined production possibility set T with costant returns assumption that is specified by the following four postulates:

- **Postulate 1 (Ray Unboundedness).** If $(X_t, Y_t) \in T$ then $(\lambda X_t, \lambda Y_t) \in T$ for all $\lambda \geq 0$.
- **Postulate 2 (Convexity).** If $(X_t, Y_t) \in T$ and $(X_u, Y_u) \in T$, then $(\lambda X_t + (1-\lambda)X_u, \lambda Y_t + (1-\lambda)Y_u) \in T$ for all $\lambda \in [0, 1]$.
- **Postulate 3 (Monotonicity).** If $(X_t, Y_t) \in T$ and $X_u \geq X_t$, $Y_u \leq Y_t$ then $(X_u, Y_u) \in T$.
- **Postulate 4 (Inclusion of Observations).** The observed $(X_j, Y_j) \in T$ for all $j = 1, \ldots, n$.
- **Postulate 5 (Minimum extrapolation).** If a production possibility set T' satisfies Postulates 1, 2, 3, and, 4 then $T \subset T'$.

The unique production possibility set with constant returns assumption determined by the above postulates is given by:

$$T = \{(X_t, Y_t)|\ X_t \geq \sum_{j=1}^{n} \lambda_j X_j,\ Y_t \leq \sum_{j=1}^{n} \lambda_j Y_j,$$

$$\lambda_j \geq 0\ ,\ j = 1, \ldots, n\}.$$

The boundary of this convex set consists of a straight line, plane, or hyperplane through the origin, as T is a convex cone that contains all of DMU's, see Figure 1 for the simpelst case of single input and single output.

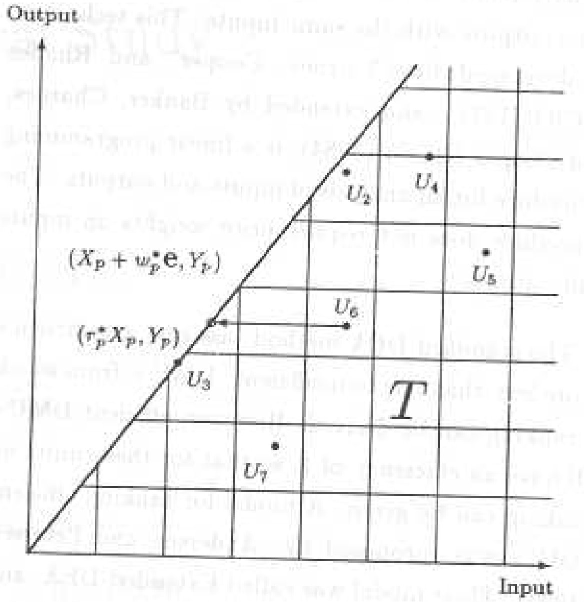


Figure 1: Production Possibility Set.

For efficiency evaluation relative to the set T, we have the following two linear programming problems:

$r_p^* = \min r_p$	$w_p^* = \min w_p$
subject to :	subject to :
$(r_p X_p, Y_p) \in T,$	$(X_p + w_p e, Y_p) \in T,$

which give the **CCR-Model and our formulation** respectively as follows:

$$r_p^* = \min r_p$$
subject to :
$$\sum_{j=1}^{n} \lambda_j X_j \leq r_p X_p,$$
$$\sum_{j=1}^{n} \lambda_j Y_j \geq Y_p,$$
$$\lambda_j \geq 0,\ j = 1, \ldots, n,$$

$w_p^* = \min w_p$
subject to :
$\sum_{j=1}^{n} \lambda_j X_j \le X_p + w_p e,$
$\sum_{j=1}^{n} \lambda_j Y_j \ge Y_p,$
$\lambda_j \ge 0, \ j = 1, \ldots, n,$

is a vector of units.

new model assigns negative efficiency scores cient units, and zero efficiency scores to all units. An extension of this model can be r ranking efficient units. This extension is l with the model except that the unit under on is excluded. The extended model is as

$$w_p^* = \min \ w_p$$
subject to :
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j X_j \le X_p + w_p e,$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j Y_j \ge Y_p,$$
$$\lambda_j \ge 0, \ j = 1, \ldots, n,$$

ads to the non-Archimidian infinitesimal from model as follows (this formulation will be ref- JAM-Model [1] in this paper):

$$w_p^* = \min \ w_p - \epsilon \left[\sum_{i=1}^{m} s_i + \sum_{r=1}^{s} s_r^{'}\right]$$
ubject to:
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j X_{ij} + s_i = X_{ip} + w_p, \ i = 1, \ldots, m,$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j Y_{rj} - s_r^{'} = Y_{rp}, \ r = 1, \ldots, s,$$
$$\lambda_j, s_i, s_r^{'} \ge 0, \ \forall j, \ i, \ r.$$

cient units will have a nonnegative efficiency while inefficient units will have the same neg- efficiency scores as before. Therefore, JAM- can be used for ranking both inefficient and nt units. It should be obvious that the op- objective function values for JAM-Model are

[1] hahloo, Alirezaee and Mehrabian Model.

dependent upon the units of measurement of input data, $X_j, \ j = 1, \ldots, n$. However, it is possible to obtain unit independence by normalization, as discussed later.

The unique production possibility set with variable returns assumption determined by postulates 2, 3, 4, and 5 is given by :

$$T = \{(X_t, \ Y_t) | \ X_t \ge \sum_{j=1}^{n} \lambda_j X_j, \ Y_t \le \sum_{j=1}^{n} \lambda_j Y_j,$$
$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \ge 0, \ j = 1, \ldots, n\}.$$

A discussion similar to the constant returns assumption leads to the BCC-Model and our second formulation, and extension of our second formulation for ranking efficient units can be as follow:

$$z_p^* = \min \ z_p - \epsilon \left[\sum_{i=1}^{m} s_i + \sum_{r=1}^{s} s_r^{'}\right]$$
subject to:
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j X_{ij} + s_i = X_{ip} + z_p, \ i = 1, \ldots, m,$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j Y_{rj} - s_r^{'} = Y_{rp}, \ r = 1, \ldots, s,$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j = 1,$$
$$\lambda_j, \ s_i, \ s_r^{'} \ge 0, \ \forall j, \ i, \ r.$$

4 The Comparison of the Two Models.

Two models for ranking the efficient DMU's were discussed in section 2 and 3. Section 2 represented the AP-Model and section 3 represented the JAM-Model. This section compares these two models using two illustrative examples.

In an actual set of data, it is possible that one or more of the data inputs and outputs are zero. It is also possible that some data inputs and outputs are small in comparison with other inputs and outputs. In these cases, AP-Model, can not correctly evaluate the efficiency of the DMU's. If the DMU under evaluation has at least one input equal to zero, the AP-Model will be infeasible and if the DMU has at least one input which is small in comparison with other inputs, AP-Model will measure this DMU without stability.

The measure given by JAM-Model successfully evaluates the above cases, so that meaningful scores are obtained for all data.

In order to make usual scores, the scores in JAM-Model may be rescaled from $[-1,+1]$ to $[0\%, 200\%]$ so that of 0 is rescaled to 100%. A score less than 100% means that the corresponding DMU is inefficient and greater than or equal to 100% means that the corresponding DMU is efficient.

4.1 Illustrative Example 1:

Table 1 gives an example of the above cases.

	A_1	A_2	A_3	B	C	D	E
input1	2	0	.1	5	10	10	2
input2	8	8	8	5	4	6	12
output1	1	1	1	1	2	2	1
output2	2	2	2	1	1	1	2

Table 1: Comparison Test Data.

There are 5 DMU's (A, B, C, D and E) each consume two inputs to produce two outputs with constant returns assumption. In order to remove the effects of changes in units of measurement from the input data, the data must be normalized before applying the method. It can be done by dividing inputs data by the maximum input (for each input).

In this example DMU_{A_1}, DMU_{A_2} and DMU_{A_3} are compared with all other DMU's (B,C,D and E) in the following three paragraphs by CCR-Model, AP-Model and JAM-Model.

- DMU_{A_1}, is evaluated to be efficient by CCR-Model, and is evaluated as efficient by AP-Model with efficiency score equal to 147%. It is evaluated as efficient by JAM-Model with efficiency score equal to $+0.276$, which rescales to $100(1+0.276) = 127.6\%$. In this case, there is no problem.

- Consider now DMU_{A_2} which has an input equal to zero. DMU_{A_2} is evaluated to be efficient by CCR-Model, but it can not be evaluated by AP-Model. However, it is evaluated as efficient by JAM-Model with efficiency score equal to $+0.310$ which rescales to 131.0%.

- Consider now DMU_{A_3} which has an input equal to 0.1. DMU_{A_3} is evaluated as efficient by CCR-Model, and can be evaluated as efficient by AP-Model with efficiency score equal to 2000% which is unstable. It is evaluated as efficient by JAM-Model with efficiency score equal to $+0.309$ which rescales to 130.9%.

4.2 Illustrative Example 2:

A comparison of these two procedures for ranking DMU's is illustrated on the Farrell frontier. Consider the DMU's of Figure 2, each produces one output using two inputs with constant returns assumption. DMU_C is efficient and it can be evaluated by AP-Model with efficiency score of $(100\frac{OC'}{OC})$ and evaluated by JAM-Model with efficiency score w_C^* which rescales to $100(1+w_C^*)$.

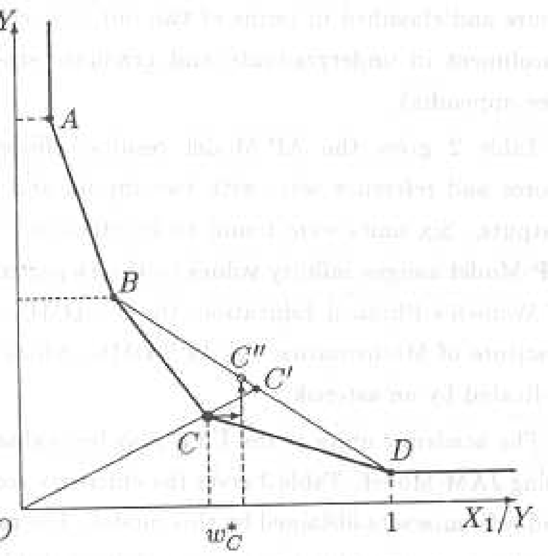


Figure 2: Farrell Efficiency Measurements.

ome examples are presented in the following fig- s that show the AP-Model cannot evaluate the ciencies of some DMU's correctly.

- DMU_A has zero for input 1 (see Figure 3):

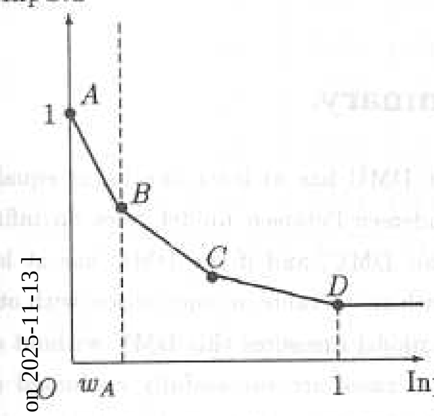


Figure 3: DMU_A has zero for input 1.

this figure AP-Model evaluates DMU_A with effi- ency score of $(100\frac{OA'}{OA})$% that is ∞ but this DMU is evaluated with ficiency score of $100(1+w_A)$% by JAM-Model.

- DMU_A has small value for input 1 (see Figure 4):

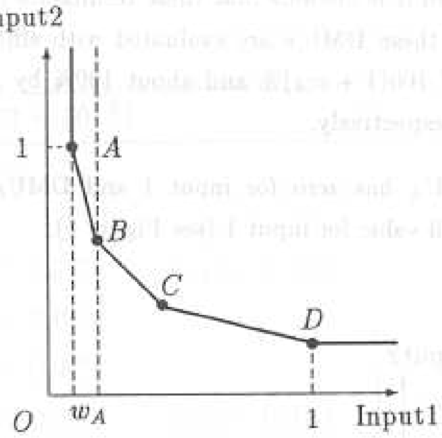


Figure 4: DMU_A has small value for input 1.

In this figure, AP-Model evaluates DMU_A with efficiency score of $(100\frac{OA'}{OA})$% that is much greater than 100%, where is unstable, but this DMU is evaluated with efficiency score of $100(1+w_A)$% by JAM-Model.

- DMU_A and DMU_B are similar units that have small values for input 1 (see Figure 5):

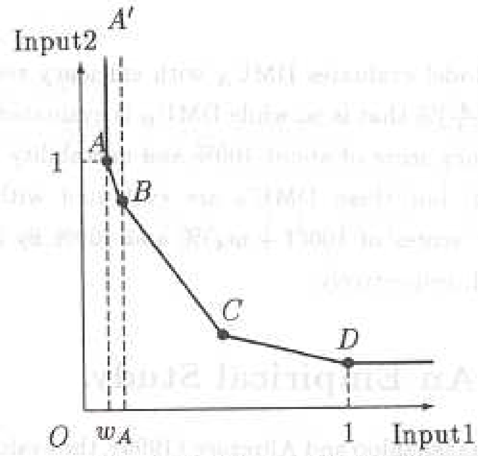


Figure 5: DMU_A and DMU_B are similar with small values for input 1.

In this figure, AP-Model evaluates DMU_A with efficiency score of much grater than 100%, while DMU_B is evaluated with efficiency score of about 100%, and it is obvious that these results are unstable, but these DMU's are evaluated with efficiency scores of $100(1+w_A)\%$ and about 100% by JAM-Model, respectively.

- DMU_A has zero for input 1 and DMU_B has small value for input 1 (see Figure 6):

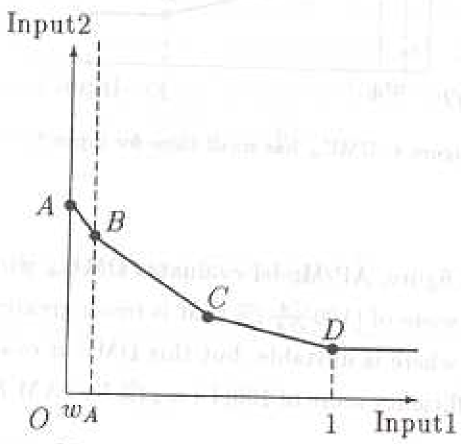


Figure 6: DMU_A has zero for input 1 and DMU_B has small value for input 1.

AP-Model evaluates DMU_A with efficiency score of $(100\frac{OA'}{OA})\%$ that is ∞ while DMU_B is evaluated with efficiency score of about 100% and unstability is observed, but these DMU's are evaluated with efficiency scores of $100(1+w_A)\%$ and 100% by JAM-Model, respectively.

5 An Empirical Study.

In Jahanshahloo and Alirezaee (1995), the evaluation of teaching in the UTE was considered. Teaching inputs were expressed in teacher hours and classified in terms of two inputs, professorial staff and instructors. Teaching outputs were expressed in student hours and classified in terms of two outputs, course enrollment in undergraduate and graduate studies (see appendix).

Table 2 gives the AP-Model results, efficiency scores and reference sets, with two inputs and two outputs. Six units were found to be efficient. The AP-Model assigns infinity values to the Department of Women's Physical Education, the 9^{th} DMU, and Institute of Mathematics, the 19^{th} DMU, which are indicated by an asterisk.

The academic units at the UTE may be evaluated using JAM-Model. Table 3 gives the efficiency scores and reference sets obtained by this model. The ranking is approximately the same as that of Table 2, except that DMU's 9 and 19 now have an explicit ranking.

Extended model, JAM-Model, successfully evaluated all efficient academic units at the UTE, in contrast to AP-Model.

In the application of AP-Model and JAM-Model on real data-set of the UTE, computational DEA issues of Ali (1994), Ali (1993), and Ali and Seiford (1993) have been considered.

6 Summary.

If an efficiet DMU has at least one input equal to zero the Andersen-Petersen model gives an infinite result for this DMU, and if the DMU has at least one input with small value in comparison with other inputs, this model measures this DMU without stability. These cases are successfully evaluated and ranked by the new model proposed in this paper.

omputational difficulties of the Andersen-Petersen model were observed in evaluating efficient academic units at the UTE. This model could not evaluate the Department of Women's Physical Education and the Institute of Mathematics. These academic units were successfully evaluated by the new model.

DMU	Eff.	Ref. Sets	$(\epsilon = 0.33 \times 10^{-6})$	
9	*			
19	*			
2	174%	$\lambda_2 = 0.492$	$\lambda_7 = 1.173$	$\lambda_{10} = 0.114$
15	133%	$\lambda_1 = 0.938$	$\lambda_{19} = 2.064$	
5	130%	$\lambda_8 = 0.956$	$\lambda_{19} = 0.479$	
1	115%	$\lambda_2 = 0.492$	$\lambda_{10} = 0.220$	$\lambda_{15} = 0.353$
8	97%	$\lambda_2 = 0.276$	$\lambda_5 = 0.648$	$\lambda_9 = 0.641$
10	96%	$\lambda_1 = 1.060$	$\lambda_2 = 0.603$	
3	95%	$\lambda_1 = 0.585$	$\lambda_2 = 0.073$	
17	89%	$\lambda_2 = 0.375$	$\lambda_5 = 0.091$	$\lambda_{19} = 0.338$
18	85%	$\lambda_2 = 0.978$	$\lambda_5 = 0.191$	$\lambda_{19} = 0.186$
7	71%	$\lambda_2 = 0.487$	$\lambda_9 = 0.204$	
12	66%	$\lambda_1 = 0.564$	$\lambda_2 = 0.285$	$\lambda_{19} = 0.392$
4	63%	$\lambda_1 = 0.542$	$\lambda_2 = 0.156$	
6	58%	$\lambda_1 = 0.131$	$\lambda_2 = 0.274$	
16	57%	$\lambda_1 = 0.231$	$\lambda_2 = 0.582$	
14	54%	$\lambda_1 = 0.726$	$\lambda_2 = 0.232$	
13	45%	$\lambda_1 = 1.210$	$\lambda_2 = 0.099$	
11	45%	$\lambda_1 = 0.048$	$\lambda_2 = 0.528$	$\lambda_{19} = 0.504$

Table 2: AP-Model Efficiency Scores for 19 Academic Units of the UTE.

DMU	Eff.	Rescaled	Ref. Sets	$(\epsilon = 0.55 \times 10^{-6})$	
19	+0.281	128%	$\lambda_{15} = 0.579$	$\lambda_{17} = 0.850$	$\lambda_{19} = 0.094$
5	+0.104	110%	$\lambda_2 = 0.033$	$\lambda_8 = 0.831$	$\lambda_{19} = 0.580$
2	+0.092	109%	$\lambda_2 = 0.575$	$\lambda_7 = 0.647$	$\lambda_{10} = 0.274$
15	+0.065	106%	$\lambda_1 = 0.938$	$\lambda_{19} = 1.491$	
1	+0.047	105%	$\lambda_2 = 0.575$	$\lambda_{10} = 0.177$	$\lambda_{15} = 0.358$
9	+0.043	104%	$\lambda_2 = 0.789$	$\lambda_9 = 0.701$	
8	-0.010	99%	$\lambda_2 = 0.228$	$\lambda_5 = 0.648$	$\lambda_9 = 0.701$
3	-0.011	99%	$\lambda_1 = 0.590$	$\lambda_2 = 0.066$	
7	-0.020	98%	$\lambda_2 = 0.310$	$\lambda_9 = 0.428$	
10	-0.021	98%	$\lambda_1 = 1.055$	$\lambda_2 = 0.609$	
17	-0.022	98%	$\lambda_2 = 0.376$	$\lambda_5 = 0.091$	$\lambda_{19} = 0.338$
18	-0.051	95%	$\lambda_2 = 0.870$	$\lambda_5 = 0.265$	$\lambda_{19} = 0.094$
6	-0.070	93%	$\lambda_1 = 0.044$	$\lambda_2 = 0.398$	
4	-0.118	88%	$\lambda_1 = 0.561$	$\lambda_2 = 0.128$	
16	-0.141	86%	$\lambda_1 = 0.007$	$\lambda_2 = 0.903$	$\lambda_{19} = 0.029$
12	-0.153	85%	$\lambda_1 = 0.717$	$\lambda_{15} = 0.049$	$\lambda_{19} = 0.278$
14	-0.235	77%	$\lambda_1 = 0.752$	$\lambda_2 = 0.194$	
11	-0.250	75%	$\lambda_1 = 0.379$	$\lambda_2 = 0.054$	$\lambda_{19} = 0.371$
13	-0.457	54%	$\lambda_1 = 1.151$	$\lambda_{15} = 0.137$	

Table 3: JAM-Model Efficiency Scores for 19 Aacademic Units of the UTE.

ent. Useful comments from Dr.
e, Professor of Economics, the Uni-
ry, Canada, Dr. R. M. Thrall, Pro-
of Administration Jones Graduate
nistration and Noah Harding Profes-
Mathematical Sciences, Rice Univer-
one anonymous referee are gratefully

es

"Streamlined Computation for Data
ent Analysis," *European J. Oper.*
3), 64(1), 61-68.

"Computational Aspects of Data En-
t Analysis, in: A. Charnes, W. W.
A. Y. Lewin, L. M. Seiford, editors,"
elopment Analysis: Theory, Methodol-
Applications, Boston: Kelevwer Aca-
blishers, (1994).

., and L. M. Seiford, "Computational
and Infinitesimals in Data Envelop-
alysis," *INFOR*, (1993), 31(4), 290-297.

n, P.; Petersen, N. C.,"A Procedure for
Efficient Units in Data Envelopment
s," *Management Scince.*, 39, (1993),
64.

R. D., A. Charnes, W. W. Cooper,
Models for Estimating Technical and
nefficiencies in Data Envelopment Anal-
Management Science, 30, (1984), 1078-

es, A., W. W. Cooper, E. Rhodes, "Mea-
the Efficiency of Decision Making Units,"
an J. Oper. Res., 2, (1978), 429-444.

es, A., W. W. Cooper, R. M. Thrall, "A
ure for Classifying and Characterizeing
nce and Inefficiency in Data Envelopment
Analysis," *The Journal of Productivity Analysis*, (1991), 2, 197-237.

[8] Jahanshahloo, G. R., M. R. Alirezaee, "Measuring the Efficiency of Academic Units at the Teacher Training University," *Proceeding of the 26th Annual Iranian Mathematics Conference*, (1995), 167-171.

[9] Thompson, R. G., P. S. Dharmapala, and R. M. Thrall, "Importance for DEA of Zeros in Data, Multipliers, and Solutions," *The Journal of Productivity Analysis*, (1993), 4, 379-390.

No.	Department/Institute	I1	I2	O1	O2
	Faculty of Literature				
1	Persian Literature	81.0	87.6	5191	205
2	Theology and Islamic Culture	85.0	12.8	3629	0
3	History	56.7	55.2	3302	0
4	Geography	91.0	78.8	3379	8
5	Foreign Languages	216.0	72.0	5368	639
6	Arabic Language and Literature	58.0	25.6	1674	0
7	Social Sciences	112.2	8.8	2350	0
	Faculty of Physical Ed.				
8	Men Physical Education	293.2	52.0	6315	414
9	Women Physical Education	186.6	0.0	2865	0
	Faculty of Sciences				
10	Mathematics	143.4	105.2	7689	66
11	Geology	108.7	127.0	2165	266
12	Bioiogy	105.7	134.4	3963	315
13	Chemistry	235.0	236.8	6643	236
14	Physics	146.3	124.0	4611	128
	Faculty of Education				
15	Foundations of Education	57.0	203.0	4869	540
16	Instructional Technology	118.7	48.2	3313	16
17	Psychology	58.0	47.4	1853	230
18	Guidance and Counseling	146.0	50.8	4578	217
19	Institute of Mathematics	0.0	91.3	0	508

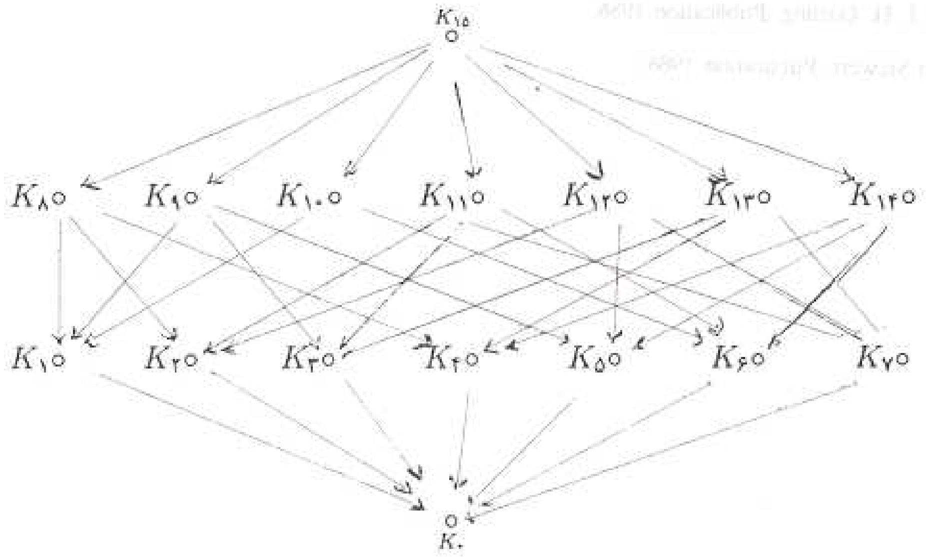
APPENDIX: Inputs and Outputs for 19 Academic Units of the UTE in the First Semester, 1993-94.

ferences:

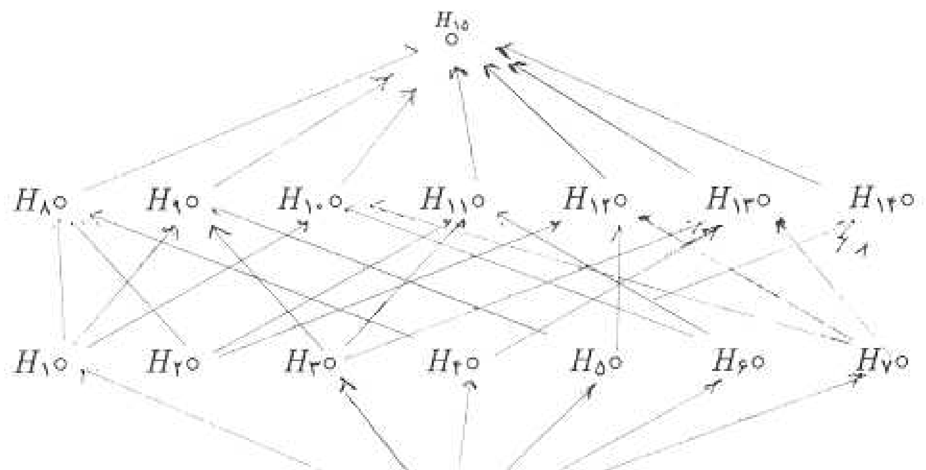
Galois Theory, Joseph Rotman, Publication 1990.

Galois Theory, D. J. H. Garling, Publication 1986.

Galois Theory, Ian Stewert, Publication 1986.



شبکه زیر میدانهای $Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$



شبکه زیر گروههای $\mathrm{Gal}_Q(f)$

برهان: بنابر بحث‌های قبل از قضیه ۴ اگر $\gamma = \sqrt{p} + \sqrt{q} + \sqrt{t}$ آنگاه $Q(\gamma) = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$. و $[Q(\gamma) : Q] = 8$ و در نتیجه چندجمله‌ای مینیمال γ روی Q از درجه ۸ می‌باشد. لذا اگر $h(x) \in Q[x]$ چندجمله‌ای ناصفر باشد به قسمی که $h(\gamma) = 0$ آنگاه $\deg(h(x)) \geq 8$.

$$
\begin{aligned}
&\gamma = \sqrt{p} + \sqrt{q} + \sqrt{t},\\
&\gamma - \sqrt{p} = \sqrt{q} + \sqrt{t},\\
&\gamma^2 + p - 2\gamma\sqrt{p} = q + t + 2\sqrt{qt},\\
&(\gamma^2 + p - q - t)^2 = 2(\gamma\sqrt{p} + \sqrt{qt}),\\
&\gamma^4 + 2(p - q - t)\gamma^2 + (p - q - t)^2 = 4(\gamma^2 p + qt + 2\gamma\sqrt{pq}),\\
&\gamma^4 - 2(p + q + t)\gamma^2 + (p^2 + q^2 + t^2 - 2pt - 2pq - 2qt) = 8\gamma\sqrt{pqt}.\\
&\gamma^8 - 4(p + q + t)\gamma^6 + [2(p^2 + q^2 + t^2 - 2pq - 2pt - 2qt) + 4(p + q + t)^2]\gamma^4\\
&\quad - [4(p + q + t)(p^2 + q^2 + t^2 - 2pq - 2pt - 2qt) + 64pqt]\gamma^2\\
&\quad + (p^2 + q^2 + t^2 - 2pq - 2pt - 2qt) = 0.
\end{aligned}
$$

بنابراین γ صفر $f(x)$ است. از این که چندجمله‌ای مینیمال γ روی Q از درجه ۸ می‌باشد نتیجه می‌شود که روی Q تحویل‌ناپذیر است. چه در غیراین صورت γ صفر یک چندجمله‌ای از درجهٔ کوچکتر از ۸ می‌باشد که غیر است.▲

با فرض $K_i = \phi(H_i)$، $(0 \leq i \leq 15)$ خواهیم داشت

$$
\begin{aligned}
&K_0 = Q(\sqrt{p}, \sqrt{q}, \sqrt{t}),\ K_1 = Q(\sqrt{q}, \sqrt{t}),\ K_2 = Q(\sqrt{p}, \sqrt{t}),\ K_3 = Q(\sqrt{p}, \sqrt{q})\\
&K_4 = Q(\sqrt{t}, \sqrt{pq}),\ K_5 = Q(\sqrt{q}, \sqrt{pt}),\ K_6 = Q(\sqrt{p}, \sqrt{qt}),\ K_7 = Q(\sqrt{pq}, \sqrt{pt}, \sqrt{qt}),\\
&K_8 = Q(\sqrt{t}),\ K_9 = Q(\sqrt{q}),\ K_{10} = Q(\sqrt{qt}),\ K_{11} = Q(\sqrt{p}),\ K_{12} = Q(\sqrt{pt})\\
&K_{13} = Q(\sqrt{pq}),\ K_{14} = Q(\sqrt{pqt}), K_{15} = Q.
\end{aligned}
$$

در صفحه بعد شبکه زیرگروههای $\mathrm{Gal}_Q(f)$ و شبکه زیرمیدانهای $Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$ را جهت مقایسه نشان می‌ده

$$[Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) : Q] = |\operatorname{Gal}_Q(f)| = ۸.$$

لذا $\operatorname{Gal}_Q(f)$ دارای ۱۶ زیرگروه به شرح زیر است:

$$\begin{aligned}
&H_۰ = \{\sigma_۰\},\ H_۱ = \{\sigma_۰, \sigma_۱\},\ H_۲ = \{\sigma_۰, \sigma_۲\},\ H_۳ = \{\sigma_۰, \sigma_۳\},\ H_۴ = \{\sigma_۰, \sigma_۴\}\\
&H_۵ = \{\sigma_۰, \sigma_۵\},\ H_۶ = \{\sigma_۰, \sigma_۶\},\ H_۷ = \{\sigma_۰, \sigma_۷\},\ H_۸ = \{\sigma_۰, \sigma_۱, \sigma_۲, \sigma_۴\}\\
&H_۹ = \{\sigma_۰, \sigma_۱, \sigma_۳, \sigma_۵\},\ H_{۱۰} = \{\sigma_۰, \sigma_۱, \sigma_۶, \sigma_۷\},\ H_{۱۱} = \{\sigma_۰, \sigma_۲, \sigma_۳, \sigma_۶\},\\
&H_{۱۲} = \{\sigma_۰, \sigma_۲, \sigma_۵, \sigma_۷\},\ H_{۱۳} = \{\sigma_۰, \sigma_۳, \sigma_۴, \sigma_۷\},\ H_{۱۴} = \{\sigma_۰, \sigma_۴, \sigma_۵, \sigma_۶\},\\
&H_{۱۵} = \operatorname{Gal}_Q(f).
\end{aligned}$$

فرض کنید $A = \{H_i | ۰ \leq i \leq ۱۵\}$ و $B = \{K | Q \leq K \leq Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$ و

$$\begin{aligned}
\psi : A &\longrightarrow B.\\
H_i &\rightsquigarrow \phi(H_i)
\end{aligned}$$

و c, b, a اعداد گویای ناصفری باشند و $\alpha = a\sqrt{p} + b\sqrt{q} + c\sqrt{t}$. دراین صورت $Q(\alpha)$ زیرمیدانی از $Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$ است و برای هر $(۱ \leq i \leq ۱۵)$، $\alpha \notin \phi(H_i)$. زیرا برای هر چنین i زیرگروه H_i عضوی دارد که α را روی خودش تصویر نمی‌کند. بنابراین $(۱ \leq i \leq ۱۵)$، $Q(\alpha) \neq \phi(H_i)$ و در نتیجه $Q(\alpha) = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$. لذا

$$[Q(\alpha) : Q] = [Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) : Q] = ۸.$$

بنابراین چندجمله‌ای مینیمال α روی Q از درجه ۸ می‌باشد، در نتیجه اگر $g(x) \in Q[x]$. یک چند جمله‌ای ناصفر باشد به قسمی که $g(\alpha) = ۰$ آنگاه $\deg(g(x)) \geq ۸$.

قضیه ۴: اگر p، q و t سه‌عدد اول دوبه‌دو متمایز باشند آنگاه

$$\begin{aligned}
f(x) = {}& x^۸ - ۴(p+q+t)x^۶ + ۲[(p+q+t)^۲ + ۲(p^۲+q^۲+t^۲)]x^۴\\
& - ۴[(p+q+t)(p^۲+q^۲+t^۲ - ۲pq - ۲pt - ۲qt) + ۱۶pqt]x^۲\\
& + (p^۲+q^۲+t^۲ - ۲pq - ۲pt - ۲qt)^۲,
\end{aligned}$$

روی Q تحویل‌ناپذیر است.

برهان: $Q(i,\sqrt{m}) = Q(i+\sqrt{m})$. لذا

$$[Q(i+\sqrt{m}):Q] = [Q(i,\sqrt{m}):Q] = ۴$$

بنابراین چندجمله‌ای مینیمال $i+\sqrt{m}$ روی Q از درجهٔ ۴ می‌باشد، در نتیجه اگر $s(x)\in Q[x]$ یک چند
باشد به قسمی که $s(i+\sqrt{m})=۰$ آنگاه $\deg(s(x))\geq ۴$. بافرض

$$\begin{aligned}
&\alpha = i+\sqrt{m},\\
&\alpha^{۲} = -۱+m+۲i\sqrt{m},\\
&\alpha^{۴}+(۱-m)^{۲}+۲(۱-m)\alpha^{۲} = -۴m,\\
&\alpha^{۴}+(۱-m)^{۲}+۲(۱-m)\alpha^{۲} = -۴m,\\
&\alpha^{۴}+۲(۱-m)\alpha^{۲}+(m+۱)^{۲} = ۰
\end{aligned}$$

بنابراین α صفر چندجمله‌ای $t(x)$ می‌باشد. در نتیجه $t(x)$ روی Q تحویل‌ناپذیر است چه در غیر این صو
یک چندجمله‌ای از درجهٔ حداکثر ۳ می‌باشد که غیرممکن است.▲

فرض کنید p، q و t سه عدد اول دوبه‌دو متمایز باشند و $f=(x^{۲}-p)(x^{۲}-q)(x^{۲}-t)$. در
$Q(\sqrt{p},\sqrt{q},\sqrt{t})/Q$ توسیع میدان تجزیه‌ای f روی Q است.

$$\overline{q},\sqrt{t} = \left\{a+b_{۱}\sqrt{p}+b_{۲}\sqrt{q}+b_{۳}\sqrt{t}+c_{۱}\sqrt{pq}+c_{۲}\sqrt{pt}+c_{۳}\sqrt{qt}+d\sqrt{pqt} \,\middle|\, a,b_i,c_i,d\in Q\right\},$$

$$\mathrm{l}_Q(f) = \{\sigma_{۰},\sigma_{۱},\sigma_{۲},\sigma_{۳},\sigma_{۴},\sigma_{۵},\sigma_{۶},\sigma_{۷}\}.$$

اگر $\sigma\in \mathrm{Gal}_Q(f)$ آنگاه برای هر $q\in Q$ داریم $\sigma(q)=q$. لذا برای مشخص کردن یک خ
$\sigma: Q(\sqrt{p},\sqrt{q},\sqrt{t})\longrightarrow Q(\sqrt{p},\sqrt{q},\sqrt{t})$ کافی است $\sigma(\sqrt{p})$، $\sigma(\sqrt{q})$ و $\sigma(\sqrt{t})$ را معین کنیم.
$\sigma(\sqrt{p})=\pm\sqrt{p}$ و $\sigma(\sqrt{q})=\pm\sqrt{q}$ و $\sigma(\sqrt{t})=\pm\sqrt{t}$، نتیجه می‌شود که ۸ عضو $\mathrm{Gal}_Q(f)$ به شرح ز

$$\begin{array}{l}\overline{p}\longrightarrow\sqrt{p}\\ \overline{q}\longrightarrow\sqrt{q}\\ \overline{t}\longrightarrow\sqrt{t}\end{array},\quad
\sigma_{۱}:\begin{cases}\sqrt{p}\longrightarrow-\sqrt{p}\\ \sqrt{q}\longrightarrow\sqrt{q}\\ \sqrt{t}\longrightarrow\sqrt{t}\end{cases},\quad
\sigma_{۲}:\begin{cases}\sqrt{p}\longrightarrow\sqrt{p}\\ \sqrt{q}\longrightarrow-\sqrt{q}\\ \sqrt{t}\longrightarrow\sqrt{t}\end{cases},\quad
\sigma_{۳}:\begin{cases}\sqrt{p}\longrightarrow\sqrt{p}\\ \sqrt{q}\longrightarrow\sqrt{q}\\ \sqrt{t}\longrightarrow-\sqrt{t}\end{cases}$$

$$\begin{array}{l}\overline{p}\longrightarrow-\sqrt{p}\\ \overline{q}\longrightarrow-\sqrt{q}\\ \overline{t}\longrightarrow\sqrt{t}\end{array},\quad
\sigma_{۵}:\begin{cases}\sqrt{p}\longrightarrow-\sqrt{p}\\ \sqrt{q}\longrightarrow\sqrt{q}\\ \sqrt{t}\longrightarrow-\sqrt{t}\end{cases};\quad
\sigma_{۶}:\begin{cases}\sqrt{p}\longrightarrow\sqrt{p}\\ \sqrt{q}\longrightarrow-\sqrt{q}\\ \sqrt{t}\longrightarrow-\sqrt{t}\end{cases},\quad
\sigma_{۷}:\begin{cases}\sqrt{p}\longrightarrow-\sqrt{p}\\ \sqrt{q}\longrightarrow-\sqrt{q}\\ \sqrt{t}\longrightarrow-\sqrt{t}\end{cases}$$

برهان: چون α به هیچیک از زیرمیدانهای $Q(\sqrt{p})$، $Q(\sqrt{q})$، $Q(\sqrt{pq})$ و Q تعلق ندارد، و $Q(\alpha)$ زیرمیدانی از $Q(\sqrt{p},\sqrt{q})$ است که با هیچیک از این چهار زیرمیدان برابر نیست پس $Q(\alpha)=Q(\sqrt{p},\sqrt{q})$.▲

قضیه ۲: اگر p, q دو عدد صحیح و مثبت و خالی از مربع باشند به قسمی که $p \nmid q$ و $q \nmid p$ و a, b اعداد گویای ناصفر باشند آنگاه چندجمله‌ای $f(x)=x^۴-۲(a^۲p+b^۲q)x^۲+(a^۲p-b^۲q)^۲$ روی Q تحویل‌ناپذیر است.

برهان: با فرض $\alpha = a\sqrt{p}+b\sqrt{q}$. بنابر لم یک داریم $Q(\alpha)=Q(\sqrt{p},\sqrt{q})$. بنابراین $[Q(\alpha):Q]=[Q(\sqrt{p},\sqrt{q}):Q]=۴$. در نتیجه چندجمله‌ای مینیمال α روی Q از درجهٔ ۴ می‌باشد. لذا اگر $g(x)\in Q[x]$ یک چندجمله‌ای ناصفر باشد به قسمی که

$$g(\alpha)=\circ \quad \text{آنگاه} \quad \deg(g(x))\geq ۴ \tag{۲}$$

$$\begin{aligned}
&\alpha = a\sqrt{p}+b\sqrt{q},\\
&\alpha^۲ = a^۲p+b^۲q+۲ab\sqrt{pq},\\
&\alpha^۲-(a^۲p+b^۲q)=۲ab\sqrt{pq},\\
&\alpha^۴+(a^۲p+b^۲q)^۲-۲(a^۲p+b^۲q)\alpha^۲=۴a^۲b^۲pq,\\
&\alpha^۴-۲(a^۲p+b^۲q)\alpha^۲+(a^۲p-b^۲q)^۲=\circ
\end{aligned}$$

بنابراین α صفر $f(x)$ می‌باشد. از این که چندجمله‌ای مینیمال α در Q از درجه ۴ می‌باشد نتیجه می‌شود که $f(x)$ روی Q تحویل‌ناپذیر است. چه در غیر این صورت α صفر یک چندجمله‌ای از درجهٔ کوچکتر از ۴ روی Q می‌باشد که با (۲) تناقض دارد.▲

نتیجه یک: اگر p, q دو عدد خالی از مربع باشند به قسمی که $p \nmid q$ و $q \nmid p$ آنگاه چندجمله‌ای $g(x)=x^۴-۲(p+q)x^۲+(p-q)^۲$ روی Q تحویل‌ناپذیر است.

برهان: با قراردادن $a=b=۱$ در قضیه ۲ نتیجه حاصل می‌شود.▲

نتیجه ۲: اگر p, q دو عدد طبیعی خالی از مربع باشند به قسمی که $p \nmid q$ و $q \nmid p$ و a یک عدد گویای ناصفر باشد آنگاه چندجمله‌ای $h(x)=x^۴-۲(p+q)a^۲x^۲+a^۴(p-q)^۲$ روی Q تحویل‌ناپذیر است.

برهان: با قراردادن $a=b$ در قضیه ۲ نتیجه حاصل می‌شود.▲

قضیه ۳: اگر m عددی صحیح و مثبت و مربع کامل نباشد (یعنی عددی اول چون p وجود دارد که $p^k|m$ و $p^{k+۱}\nmid m$ و k فرد است) آنگاه چندجمله‌ای $t(x)=x^۴-۲(m-۱)x^۲+(m+۱)^۲$ روی Q تحویل‌ناپذیر است.

یک زیرمیدان E می‌باشد که شامل F است. به‌عکس اگر K زیرمیدانی از E و شامل F ... $\{\sigma \in \mathrm{Gal}_F(f) | \forall k \in K(\sigma(k)=k)\}$ زیرگروهی از $\mathrm{Gal}_F(f)$ است. چنانچه:

$A = \{K | K$ زیرمیدانی از E که شامل F است$\}$,

$B = \{H | H$ یک زیرگروه $\mathrm{Gal}_F(f)$ است$\}$,

آنگاه تابع (۱) $\psi : B \longrightarrow A$ با ضابطهٔ $\psi(H) = \phi(H)$ یک تناظر یک‌به‌یک است.

فرض کنید p, q دو عدد خالی از مربع باشند به قسمی که $p \neq (p,q) \neq q$ و $f = (x^2 - p)(x^2 - q)$ ... این صورت $Q(\sqrt{p}, \sqrt{q})/Q$ یک توسیع میدان تجزیه‌ای f است.

$$Q(\sqrt{p}, \sqrt{q}) = \{a_0 + a_1\sqrt{p} + a_2\sqrt{q} + a_3\sqrt{pq} | a_i \in Q\}$$

$$|\mathrm{Gal}_Q(f)| = [Q(\sqrt{p}, \sqrt{q}) : Q] = 4.$$

با فرض $\mathrm{Gal}_Q(f) = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$، هر یک از σ_i، $(0 \leq i \leq 3)$ یک خودریختی روی $Q(\sqrt{p}, \sqrt{q})$ ... که هر عضو Q را ثابت نگهمیدارند. برای مشخص کردن یک خودریختی روی $Q(\sqrt{p}, \sqrt{q})$ کافی است $\sigma(\sqrt{p})$ ... $\sigma(\sqrt{q})$ را معین نمائیم. از آنجا که $\sigma(\sqrt{p}) = \pm\sqrt{p}$ و $\sigma(\sqrt{q}) = \pm\sqrt{q}$، نتیجه می‌شود که اعضای $\mathrm{Gal}_Q(f)$ ... شرح زیر می‌باشند

$$\sigma_0 : \begin{cases} \sqrt{p} \longrightarrow \sqrt{p} \\ \sqrt{q} \longrightarrow \sqrt{q} \end{cases}, \quad \sigma_1 : \begin{cases} \sqrt{p} \longrightarrow -\sqrt{p} \\ \sqrt{q} \longrightarrow \sqrt{q} \end{cases}, \quad \sigma_2 : \begin{cases} \sqrt{p} \longrightarrow \sqrt{p} \\ \sqrt{q} \longrightarrow -\sqrt{q} \end{cases}, \quad \sigma_3 : \begin{cases} \sqrt{p} \longrightarrow -\sqrt{p} \\ \sqrt{q} \longrightarrow -\sqrt{q} \end{cases}$$

بنابراین $\mathrm{Gal}_Q(f)$ دارای زیرگروههای

$$H_0 = \{\sigma_0\}, \quad H_1 = \{\sigma_0, \sigma_1\}, \quad H_4 = \mathrm{Gal}_Q(f), \quad H_3 = \{\sigma_0, \sigma_3\}, \quad H_2 = \{\sigma_0, \sigma_2\}.$$

در نتیجه زیرمیدانهای متناظر H_iها بنابر رابطه (۱) که تمام زیرمیدانهای $Q(\sqrt{p}, \sqrt{q})$ نیز می‌باشند به شرح زیر می‌باشند

$$\phi(H_0) = Q(\sqrt{p}, \sqrt{q})$$

$$\phi(H_4) = Q, \quad \phi(H_1) = Q(\sqrt{q}), \quad \phi(H_2) = Q(\sqrt{p}), \quad \phi(H_3) = Q(\sqrt{pq})$$

لم ۱: اگر a, b اعداد گویای ناصفر و p, q دو عدد خالی از مربع باشند به قسمی که $p \neq (p,q) \neq q$ و $\alpha = a\sqrt{p} + b\sqrt{q}$ آنگاه $Q(\alpha) = Q(\sqrt{p}, \sqrt{q})$.