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# Arens Regularity and Factorization Property

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## Abstract

In this paper, we study the Arens regularity properties of module actions and we extend some proposition from Baker, Dales and Lau into general situations. We establish some relationships between topological centers of module actions and factorization properties of them with some results in group algebras.

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## Preliminaries and Introduction

In 1951 Arens shows that the second dual $A^{**}$ of Banach algebra $A$ endowed with the either Arens multiplications is a Banach algebra, see [1]. Some authors have some works in this area. The most important papers in this subject are some papers of Dales, Laali, Arikan, Ebrahimi Vishki, Pim, Eshaghi Gordji and Filali. The constructions of the two Arens multiplications in $A^{**}$ lead us to definition of topological centers for $A^{**}$ with respect to both Arens multiplications. The topological centers of Banach algebras, module actions and applications of them were introduced and discussed in [3], [5], [6], [9], [10], [11], [13], [15], [16], [17], [18], [19], [24], [25]. In this paper, we extend some problems from [3], [5], [6], [9], [11], [16], [22] to the general criterion on module actions that has been studied by Dales, Laali, Vishki, Eshaghi and Filali with some applications in group algebras.

Now we introduce some notations and definitions that we use in this paper.

Throughout this paper, $A$ is a Banach algebra and $A^{*}$, $A^{**}$, respectively, are the first and second dual of $A$. For $a \in A$ and $a' \in A^{*}$, we denote by $a'a$ and $aa'$ respectively,

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the functionals on $A^*$ defined by $\langle a'a,b\rangle = \langle a', ab\rangle = a'(ab)$ and $\langle aa', b\rangle = \langle a', ba\rangle = a'(ba)$ for all $b \in A$. The Banach algebra $A$ is embedded in its second dual via the identification $\langle a, a'\rangle = \langle a', a\rangle$ for every $a \in A$ and $a' \in A^*$. We denote the set $\{a'a : a \in A \text{ and } a' \in A^*\}$ and $\{aa' : a \in A \text{ and } a' \in A^*\}$ by $A^*A$ and $AA^*$, respectively, clearly these two sets are subsets of $A^*$.The extension of bilinear maps on normed space and the concept of regularity of bilinear maps were studied by [1, 2, 5, 6, 9]. We start by recalling these definitions as follows, see [9].

Let $X, Y, Z$ be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions $m^{***}$ and $m^{t***t}$ of $m$ from $X^{**} \times Y^{**}$ into $Z^{**}$ as following

1. $m^* : Z^* \times X \to Y^*$, given by $\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $\langle m^{**}(y'', z'), x\rangle = \langle y'', m^*(z', x)\rangle$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $\langle m^{***}(x'', y''), z'\rangle = \langle x'', m^{**}(y'', z')\rangle$

where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.

The mapping $m^{***}$ is the unique extension of $m$ such that $x'' \to m^{***}(x'', y'')$ from $X^{**}$ into $Z^{**}$ is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - to - weak^*$ continuous from $Y^{**}$ into $Z^{**}$ unless $x'' \in X$. Hence the first topological center of $m$ may be defined as following

$Z_1(m) = \{x'' \in X^{**} : y'' \to m^{***}(x'', y'') \text{ is } weak^* - to - weak^* \text{ continuous}\}$.

Let now $m^t : Y \times X \to Z$ be the transpose of $m$ defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then $m^t$ is a continuous bilinear map from $Y \times X$ to $Z$, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to $m^{***}$, see [1], if $m^{***} = m^{t***t}$, then $m$ is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from $X^{**}$ into $Z^{**}$ is not in general $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of $m$ as

$Z_2(m) = \{y'' \in Y^{**} : x'' \to m^{t***t}(x'', y'') \text{ is } weak^* - to - weak^* \text{ continuous}\}$.

It is clear that $m$ is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens



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regularity of $m$ is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$ , $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [5].

The mapping $m$ is left strongly Arens irregular if $Z_1(m) = X$ and $m$ is right strongly Arens irregular if $Z_2(m) = Y$.

Let now $B$ be a Banach $A-bimodule$, and let

$$\pi_\ell : A\times B \to B \quad and \quad \pi_r : B\times A \to B.$$

be the left and right module actions of $A$ on $B$, respectively. Then $B^{**}$ is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{***} : A^{**}\times B^{**} \to B^{**} \quad and \quad \pi_r^{***} : B^{**}\times A^{**} \to B^{**}.$$

Similarly, $B^{**}$ is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{t***t} : A^{**}\times B^{**} \to B^{**} \quad and \quad \pi_r^{t***t} : B^{**}\times A^{**} \to B^{**}.$$

We may therefore define the topological centers of the left and right module actions of $A$ on $B$ as follows:

$$Z_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} : the\ map\ b'' \to \pi_\ell^{***}(a'', b'') : B^{**} \to B^{**}$$
$$is\ \ weak^*-to-weak^*\ continuous\}.$$
$$Z^t_{B^{**}}(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} : the\ map\ b'' \to \pi_r^{t***}(a'', b'') : B^{**} \to B^{**}$$
$$is\ \ weak^*-to-weak^*\ continuous\}.$$
$$Z_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} : the\ map\ a'' \to \pi_r^{***}(b'', a'') : A^{**} \to B^{**}$$
$$is\ \ weak^*-to-weak^*\ continuous\}.$$
$$Z^t_{A^{**}}(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} : the\ map\ a'' \to \pi_\ell^{t***}(b'', a'') : A^{**} \to B^{**}$$
$$is\ \ weak^*-to-weak^*\ continuous\}.$$

We note also that if $B$ is a left(resp. right) Banach $A-module$ and $\pi_\ell : A\times B \to B$ (resp. $\pi_r : B\times A \to B$) is left (resp. right) module action of $A$ on $B$, then $B^*$ is a right (resp. left) Banach $A-module$.

We write $ab = \pi_\ell(a,b)$, $ba = \pi_r(b,a)$, $\pi_\ell(a_1a_2, b) = \pi_\ell(a_1, a_2b)$,

$$\pi_r(b, a_1a_2) = \pi_r(ba_1, a_2),\ \ \pi_\ell^*(a_1b', a_2) = \pi_\ell^*(b', a_2a_1),\ \ \pi_r^*(b'a, b) = \pi_r^*(b', ab),$$

for all $a_1, a_2, a \in A$, $b \in B$ and $b' \in B^*$ when there is no confusion.

Regarding $A$ as a Banach $A-bimodule$, the operation $\pi : A\times A \to A$ extends to $\pi^{***}$ and $\pi^{t***t}$ defined on $A^{**}\times A^{**}$. These extensions are known, respectively, as the



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first(left) and the second (right) Arens products, and with each of them, the second dual space $A^{**}$ becomes a Banach algebra. In this situation, we shall also simplify our notations, see [16]. So the first (left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle,$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle.$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$\langle aoa', b\rangle = \langle a', ba\rangle,$$
$$\langle a'oa'', a\rangle = \langle a'', aoa'\rangle,$$
$$\langle a''ob'', a'\rangle = \langle b'', a'ob''\rangle.$$

for all $a, b \in A$ and $a' \in A^*$.

The regularity of a normed algebra $A$ is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let $a''$ and $b''$ be elements of $A^{**}$, the second dual of $A$. By *Goldstine's* Theorem [4, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $A$ such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta \langle a', \pi(a_\alpha, b_\beta)\rangle = \langle a''b'', a'\rangle$$

and

$$\lim_\beta \lim_\alpha \langle a', \pi(a_\alpha, b_\beta)\rangle = \langle a''ob'', a'\rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of $A^{**}$, respectively, see [16, 20].

We find the usual first and second topological center of $A^{**}$, which are

$$Z_1(A^{**}) = Z_{A^{**}}(A^{**}) = Z(\pi) = \{a'' \in A^{**} : b'' \to a''b'' \text{ is } weak^* - to - weak^* \text{ continuous}\},$$

$$Z_2(A^{**}) = Z^t_{A^{**}}(A^{**}) = Z(\pi^t) = \{a'' \in A^{**} : a'' \to a''ob'' \text{ is } weak^* - to - weak^* \text{ continuous}\}.$$

Recall that a left approximate identity (= *LAI*) [resp. right approximate identity



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$(= RAI)$ ] in Banach algebra $A$ is a net $(e_\alpha)_{\alpha\in I}$ in $A$ such that $e_\alpha a \to a$ [resp. $ae_\alpha \to a$ ]. We say that a net $(e_\alpha)_{\alpha\in I} \subseteq A$ is a approximate identity $(= AI)$ for $A$ if it is $LAI$ and $RAI$ for $A$. If $(e_\alpha)_{\alpha\in I}$ in $A$ is bounded and $AI$ for $A$, then we say that $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity ( = $BAI$ ) for $A$. Let $A$ have a $BAI$. If the equality $A^*A = A^*$, $(AA^* = A^*)$ holds, then we say that $A^*$ factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that $A^*$ factors on both sides.

An element $e''$ of $A^{**}$ is said to be a mixed unit if $e''$ is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, $e''$ is a mixed unit if and only if, for each $a'' \in A^{**}$, $a''e'' = e''oa'' = a''$. By [4, p.146], an element $e''$ of $A^{**}$ is mixed unit if and only if it is a $weak^*$ cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in $A$.

## Factorization property and topological centers

For a Banach algebra $A$ with bounded right approximate identity, Baker, Lau and Pym (See [3]) proved that $(A^*A)^\perp$ is an ideal of right annihilators in $A^{**}$ and $A^{**} \cong (A^*A)^* \oplus (A^*A)^\perp$. For a Banach $A-bimodule$ $B$, we study the similar discussion on the module actions and we show that

$$B^{**} = (B^*A)^* \oplus (B^*A)^\perp.$$

**Theorem 2.1.** Let $B$ be a Banach $A-bimodule$ and $A$ have a $BRAI$. Then the following assertions are hold:

i) $(B^*A)^\perp = \{b'' \in B^{**} : \pi_\ell^{***}(a'', b'') = 0 \ \ for\ all\ a'' \in A^{**}\}$.

ii) $(B^*A)^*$ is isomorphism with $Hom_A(B^*, A^*)$ of continuous homomorphisms of $A-$ module $B^*$ into $A-$module $A^*$.

*Proof.* i) Let $b'' \in (B^*A)^\perp$. Then for all $b' \in B^*$ and $a \in A$, we have

$$\langle \pi_\ell^{**}(b'', b'), a\rangle = \langle b'', \pi_\ell^*(b', a)\rangle = 0,$$

it follows that for all $a'' \in A^{**}$, we have

$$\langle \pi_\ell^{***}(a'', b''), b'\rangle = \langle a'', \pi_\ell^{**}(b'', b')\rangle = 0.$$

Conversely, let $b'' \in B^{**}$ such that $\pi_\ell^{***}(a'', b'') = 0$ for all $a'' \in A^{**}$. Then for all $a \in A$ and $b' \in B^*$, we have

$$\langle b'', \pi_\ell^*(b', a)\rangle = \langle \pi_\ell^{**}(b'', b'), a\rangle = \langle a, \pi_\ell^{**}(b'', b')\rangle = \langle \pi_\ell^{***}(a\ , b''), b'\rangle = 0,$$

which implies that $b'' \in (B^*A)^\perp$.



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ii) Suppose that $b'' \in B^{**}$. We define $T_{b''} \in Hom_A(B^*, A^*)$, that is, $T_{b''}b' = \pi_\ell^{**}(b'', b')$. Then $\Lambda = b'' \to T_{b''}$ is linear continuous map from $B^{**}$ into $Hom_A(B^*, A^*)$ such that

$$Ker\Lambda = \{b'' \in B^{**} : \pi_\ell^{**}(b'', b') = 0 \ \ for\ all\ b' \in B^*\}.$$

Consequently, $b'' \in Ker\Lambda$ if and only if

$$\langle b'', \pi_\ell^*(b', a) = \langle \pi_\ell^{**}(b'', b'), a\rangle = 0,$$

where $b' \in B^*$ and $a \in A$. It follows that $b'' \in (B^*A)^\perp$. Since $(B^*A)^* \cong \frac{B^{**}}{(B^*A)^\perp}$, the continuous linear mapping $\Lambda$ from $(B^*A)^*$ into $Hom_A(B^*, A^*)$ is injective.

Conversely, suppose that $T \in Hom_A(B^*, A^*)$ and $e'' \in A^{**}$ is any right identity for $A^{**}$ that we set it fixed. We define $b_T'' \in B^{**}$ such that for all $b'$, we have

$$\langle b_T'', b'\rangle = \langle e'', Tb'\rangle.$$

It is clear that the linear mapping $T \to b_T''$ is continuous. For all $a \in A$, we have

$$\langle \pi_\ell^{**}(b_T'', b'), a\rangle = \langle b_T'', \pi_\ell^*(b', a)\rangle = \langle e'', T\pi_\ell^*(b', a)\rangle = \langle e'', (Tb')a\rangle$$
$$= \langle ae'', Tb'\rangle = \langle Tb', a\rangle.$$

Consequently, $\pi_\ell^{**}(b_T'', b') = Tb'$. It follows that the linear mapping $T \to b_T'' \to T_{b_T''}$ is the identity map and consequently the isomorphism between $Hom_A(B^*, A^*)$ and $(B^*A)^*$ is established.

By using proceeding theorem we observe that $Hom_A(B^*, A^*)$ has an right identity. Also if $B = A$, we obtain Theorem 1-1 from [3].

**Corollary 2.2.** Let $B$ be a Banach $A-bimodule$ with $BRAI$ for $A$ and let $e''$ be any right identity of $A^{**}$. Then $e''B^{**} \cong (B^*A)^*$ and $(B^*A)^\perp = \{b'' - e''b'' : b'' \in B^{**}\}$. Thus $B^{**} = (B^*A)^* \oplus (B^*A)^\perp$.

**Example 2.3.** Let $G$ be an unimodular locally compact group. Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then by using Theorem 2.1, we conclude that

$$(L^p(G) * L^1(G))^\perp = \{b \in L^q(G) : a''b = 0 \ for\ every\ a'' \in (L^\infty(G))^*\},$$
$$(L^p(G) * L^1(G))^* \cong Hom_{L^1(G)}(L^p(G), L^\infty(G)).$$

**Theorem 2.4.** Assume that $B$ is a left Banach $A-module$ and $A$ has a $BAI$. Then we have the following assertions.



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1) If $B^*$ factors on the left and $B^{**}$ has a left unit as $A^{**}-module$, then $(B^*)^{\perp}=0$

2) If $B^*$ not factors on the left and $e''$ is a left unit as $A^{**}-module$ in $B^{**}$, then $e''\notin Z^{\ell}_{B^{**}}(A^{**})$.

*Proof.*

1) Let $a\in A$, $b'\in B^*$ and $b''\in(B^*A)^{\perp}$. Then

$$\langle\pi^{**}_{\ell}(b'',b'),a\rangle=\langle b'',\pi^{*}_{\ell}(b',a)\rangle=0.$$

Thus, for all $a''\in A^{**}$, we have

$$\langle\pi^{***}_{\ell}(a'',b''),b'\rangle=\langle a'',\pi^{**}_{\ell}(b'',b')\rangle=0.$$

It follows that $\pi^{***}_{\ell}(a'',b'')=0$.

Now let $e''\in A^{**}$ be a left unit as $A^{**}-module$ for $B^{**}$, then we have

$$b''=\pi^{***}_{\ell}(e'',b'')=0.$$

Thus the proof of hart (1) is complate.

2) Assume a contradiction that $e''\in Z_{B^{**}}(A^{**})$. Take $b''\in(B^*A)^{\perp}$. Then

$$b''=\pi^{***}_{\ell}(e'',b'')=\pi^{t^{***}t}_{\ell}(e'',b'').$$

Let $(e_\alpha)_\alpha\subseteq A$ such that $e_\alpha\xrightarrow{w^*}e''$ and let $b\in B$, $b'\in B^*$. Then

$$\langle\pi^{t^{**}}_{\ell}(e'',b'),b\rangle=\langle e'',\pi^{t^{*}}_{\ell}(b',b)\rangle=lim_\alpha\langle\pi^{t^{*}}_{\ell}(b',b),e_\alpha\rangle$$
$$=lim_\alpha\langle b',\pi^{t}_{\ell}(b,e_\alpha)\rangle=lim_\alpha\langle b',\pi_{\ell}(e_\alpha,b)\rangle=lim_\alpha\langle\pi^{*}_{\ell}(b',e_\alpha),b\rangle$$

It follows that

$$w^*-lim_\alpha\pi^{*}_{\ell}(b',e_\alpha)=\pi^{t^{**}}_{\ell}(e'',b').$$

Let $(b_\beta)_\beta\subseteq B$ such that $b_\beta\xrightarrow{w^*}b''$. Then

$$\langle b'',b'\rangle=\langle\pi^{t^{***t}}_{\ell}(e'',b''),b'\rangle=\langle\pi^{t^{***}}_{\ell}(b'',e''),b'\rangle=\langle b'',\pi^{t^{**}}_{\ell}(e'',b')\rangle$$
$$=lim_\beta\langle\pi^{t^{**}}_{\ell}(e'',b'),b_\beta\rangle=lim_\beta lim_\alpha\langle\pi^{*}_{\ell}(b',e_\alpha),b_\beta\rangle$$
$$=lim_\beta lim_\alpha\langle b',\pi_{\ell}(e_\alpha,b_\beta)\rangle=lim_\alpha lim_\beta\langle b',\pi_{\ell}(e_\alpha,b_\beta)\rangle$$
$$=lim_\alpha\langle b',\pi^{***}_{\ell}(e_\alpha,b'')\rangle=lim_\alpha\langle\pi^{***}_{\ell}(e_\alpha,b''),b'\rangle$$
$$=lim_\alpha\langle e_\alpha,\pi^{**}_{\ell}(b'',b')\rangle=lim_\alpha\langle b'',\pi^{*}_{\ell}(b',e_\alpha)\rangle=0.$$

It follows that $(B^*A)^{\perp}=0$. By using Corollary 2.2, we have $B^{**}=(B^*A)^{\perp}\oplus(B^*A)^*$, and so $B^{**}=(B^*A)^*$. Now since $B^*A\neq B^*$, by Hahn Banach theorem, there is $0\neq b''\in B^{**}$ such that $b''|_{B^*A}=0$. It follows that $b''\in(B^*A)^{\perp}$ which is a contradiction.

**Corollary 2.5.** For a left Banach $A-module$ $B$, we have the following statements.



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1. If $\overline{B^*A} = B^*$ and $B^{**}$ has a left unit as $A^{**} - module$, then $(B^*)^{\perp} = 0$.
2. If $\overline{B^*A} \neq B^*$ and $e''$ is a left unit as $A^{**} - module$ in $B^{**}$, then $e'' \notin Z^{\ell}_{B^{**}}(A^{**})$.

Proof is similar to Theorem 2.4.

**Corollary 2.6.** Assume that $B$ is a left Banach $A-module$ and $B^{**}$ has a left unit $A^{**} - module$. If $\overline{B^*A} \neq B^*$, then $Z^{\ell}_{B^{**}}(A^{**}) \neq A^{**}$.

In the proceeding corollary, take $B = A$ and assume that $A^*$ not factors on the left. Then, if $A$ has a $LBAI$, we conclude that $A$ is not Arens regular.

**Example 2.7.** Let $G$ be an infinite locally compact group, by using Corollary 2.6, we have the following inequality

$$Z^{\ell}_{L^1(G)^{**}}(M(G)^{**}) \neq M(G)^{**} \quad , \quad Z^{\ell}_{M(G)^{**}}(L^1(G)^{**}) \neq L^1(G)^{**}.$$

Thus $L^1(G)$ and $M(G)$ are not Arens regular.

Throughout this paper, the notations $WSC$ is used for weakly sequentially complete Banach space $A$, that is, $A$ is said to be weakly sequentially complete, if every weakly Cauchy sequence in $A$ has a weak limit in $A$.

Assume that $B$ is a Banach $A-bimodule$. We say that $B$ factors on the left (right) with respect to $A$ if $B = BA\,(B = AB)$. We say that $B$ factors on both sides, if $B = BA = AB$.

Suppose that $A$ is a Banach algebra and $B$ is a Banach $A-bimodule$. According to [28] $B^{**}$ is a Banach $A^{**} - bimodule$, where $A^{**}$ is equipped with the first Arens product. We define $B^*B$ as a subspace of $A^*$, that is, for all $b' \in B^*$ and $b \in B$, we define

$$\langle b'b, a\rangle = \langle b', ba\rangle;$$

We similarly define $B^{***}B^{**}$ as a subspace of $A^{**}$. We take $A^{(0)} = A$ and $B^{(0)} = B$.

Let $B$ be a left Banach $A-module$ and $(e_\alpha)_\alpha \subseteq A$ be a LAI [resp. weakly left approximate identity(=WLAI)] for $A$. We say that $(e_\alpha)_\alpha$ is left approximate identity ( $= LAI$ )[ resp. weakly left approximate identity (=$WLAI$ )] for $B$, if for all $b \in B$, we have $\pi_\ell(e_\alpha, b) \to b$ ( resp. $\pi_\ell(e_\alpha, b) \overset{w}{\to} b$). The definition of the right approximate identity ( = $RAI$ )[ resp. weakly right approximate identity ( = $WRAI$ )] is similar.

We say that $(e_\alpha)_\alpha$ is a approximate identity ( = $AI$ )[ resp. weakly approximate identity ($WAI$ )] for $B$, if $B$ has left and right approximate identity [ resp. weakly left and right



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approximate identity ] that are equal.

Let $B$ be a left Banach as $A-module$ and $e$ be a left unit element of $A$. Then we say that $e$ is a left unit (resp. weakly left unit) as $A-module$ for $B$, if $\pi_\ell(e,b)=b$ (resp. $\langle b', \pi_\ell(e,b)\rangle = \langle b', b\rangle$ for all $b' \in B^*$) where $b \in B$. The definition of right unit (resp. weakly right unit) as $A-module$ is similar.

We say that a Banach $A-bimodule$ $B$ is an unital, if $B$ has the same left and right unit as $A-module$.

**Lemma 2.8.** Let $B$ be a Banach $A-bimodule$. Suppose that $A$ has a $BAI$ $(e_\alpha)_\alpha \subseteq A$. Then

1) $B$ factors on the left if and only if $\pi_r(b,e_\alpha) \overset{w}{\to} b$ for every $b \in B$.
2) $B$ factors on the right if and only if $\pi_\ell(e_\alpha,b) \overset{w}{\to} b$ for every $b \in B$.
3) If $B^*$ factors on the right, then $\pi_r(b,e_\alpha) \overset{w}{\to} b$ for every $b \in B$.

*Proof.*

1) Suppose that $B$ factors on the left. Then for every $b \in B$, there are $y \in B$ and $a \in A$ such that $b = ya$. Thus for every $b' \in B^*$, we have

$$\langle b', \pi_r(b,e_\alpha)\rangle = \langle b', \pi_r(ya,e_\alpha)\rangle = \langle b', \pi_r(y,ae_\alpha)\rangle = \langle \pi_r^*(b',y), ae_\alpha\rangle$$
$$\to \langle \pi_r^*(b',y), a\rangle = \langle b', ya\rangle = \langle b', b\rangle.$$

It follows that $\pi_r(b,e_\alpha) \overset{w}{\to} b$.

Conversely, by *Cohen's factorization Theorem*, since $BA$ is a closed subspace of $B$, the result follows.

2) Proof is similar to (1).

3) Assume that $B^*$ factors on the right with respect to $A$. Then for every $b' \in B^*$, there are $y' \in B$ and $a \in A$ such that $b' = ay'$. Consequently for every $b \in B$, we have

$$\langle b', \pi_r(b,e_\alpha)\rangle = \langle ay', \pi_r(b,e_\alpha)\rangle = \langle y', \pi_r(b,e_\alpha)a\rangle = \langle y', \pi_r(b,e_\alpha a)\rangle$$
$$= \langle \pi_r^*(y',b), e_\alpha a\rangle \to \langle \pi_r^*(y',b), a\rangle = \langle y', \pi_r(b,a)\rangle = \langle ay', b\rangle$$
$$= \langle b', b\rangle.$$

It follows that $\pi_r(b,e_\alpha) \overset{w}{\to} b$.

In the proceeding theorem, if we take $B = A$, then we obtain Lemma 2.1 from [16].

**Theorem 2.9.** Let $B$ be a Banach $A-bimodule$ and $A$ has a sequential $WBAI$. Then



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we have the following assertions.

(i) Let $B^*$ be a *WSC* and $A^*$ factors on the left. Then

1. If $B$ factors on the right, it follows that $B^*$ factors on the left.
2. If $B^*$ factors on the right, it follows that $B$ factors on the left.

(ii) Let $B^{**}B^* = A^{**}A^*$. Then $A^*$ factors on the left if and only if $B^*$ factors on the left.

(iii) Suppose that $A$ is *WSC* and $B$ factors on the left (resp. right). If $B^*B = A^*$, then we have the following assertions.

1. $A$ is an unital and $B$ has a right (resp. left) unit as Banach $A-module$.
2. $A^*$ factors on the both side and $B^*$ factors on the right (resp. left).
3. $B^{**} \cong (AB^*)^*$ (resp. $B^{**} \cong (B^*A)^*$).

*Proof.* i) 1) Assume that $b'' \in B^{**}$ and $b' \in B^*$. Since $A^*$ factors on the left, there are $a' \in A^*$ and $a \in A$ such that $b''b' = a'a$. Suppose that $(e_n)_n \subseteq A$ is a sequential *WBAI* for $A$. Then we have

$$\langle b'', b'e_n \rangle = \langle b''b', e_n \rangle = \langle a'a, e_n \rangle = \langle a', ae_n \rangle \to \langle a', a \rangle.$$

It follows that the sequence $(b'e_n)_n$ is weakly Cauchy sequence in $B^*$. Since $B^*$ is *WSC*, there is $x' \in B^*$ such that $b'e_n \xrightarrow{w} x'$. On the other hand, since $B$ factors on the right, by using Lemma 2.8, for each $b \in B$, we have $e_n b \xrightarrow{w} b$. Then we have

$$\langle x', b \rangle = \lim_n \langle b'e_n, b \rangle = \lim_n \langle b', e_n b \rangle = \langle b', b \rangle.$$

It follows that $x' = b'$, and so by Lemma 2.8, $B^*$ factors on the left.

i) 2) Proof is similar to part (1).

ii) Let $a'' \in A^{**}$ and $a' \in A^*$. Then there are $b'' \in B^{**}$ and $b' \in B^*$ such that $b''b' = a''a'$. Then

$$\langle a'', a'e_n \rangle = \langle a''a', e_n \rangle = \langle b''b', e_n \rangle = \langle b'', b'e_n \rangle.$$

Thus, by *Cohen's* factorization theorem, the result follows.

iii) 1) Suppose that $(e_k)_k \subseteq A$ is a sequential *WBAI* for $A$. Let $a' \in A^*$. Since $B^*B = A^*$, there are $b' \in B^*$ and $b \in B$ such that $b'b = a'$. Since $B$ factors on the left, there are $y \in B$ and $a \in A$ such that $b = ya$. Then we have

$$\begin{aligned} \langle a', e_k \rangle &= \langle b'b, e_k \rangle = \langle b', be_k \rangle = \langle b', yae_k \rangle \\ &= \langle b'y, ae_k \rangle \to \langle b'y, a \rangle = \langle b', ya \rangle \\ &= \langle b', b \rangle. \end{aligned}$$



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This shows that the sequence $(e_k)_k \subseteq A$ is weakly sequence in $A$. Since $A$ is WSC, it convergence weakly to some element $e$ of $A$. Then, for each $x \in A$, we have

$$xe = x(w-\lim_k e_k) = w-\lim_k xe_k = a.$$

It is similar to that $ex = x$, and so $A$ is unital.

Now let $b \in B$, then

$$\langle b', be\rangle = \langle b'b, e\rangle = \lim_k \langle b'b, e_k\rangle = \lim_k \langle b', yae_k\rangle$$
$$= \lim_k \langle b'y, ae_k\rangle \to \langle b'y, a\rangle = \langle b', b\rangle.$$

Thus $be = b$ for all $b \in B$.

iii) 2) By using part (1) and [16, Theorem 2.6], it is clear that $A^*$ factors on the both side. Now let $b' \in B^*$ and $b \in B$. By part (1), set $e \in A$ as a left unite element of $B$. Then

$$\langle eb', b\rangle = \langle b', be\rangle = \langle b', b\rangle.$$

It follows that $eb' = b'$. Thus $B^*$ factors on the right.

iii) 3) Now let $b'' \in (AB^*)^\perp$. By using part (2), since $B^*$ factors on the right, for every $b' \in B^*$ there are $x' \in B^*$ and $a \in A$ such that $b' = ax'$. Then

$$\langle b'', b'\rangle = \langle b'', ax'\rangle = 0.$$

It follows that $b'' = 0$ and $(AB^*)^\perp = \{0\}$. Therefore by using Corollary 2.2, we are done.

The following conclusions hold when in the proceeding theorem, we set $A = B$.

Let $A$ has a sequential $WBAI$ and Suppose that $A$ is a $WSC$. Then

1. $A^*$ factors on the left (resp. right) if and only if $A^*$ factors on the right (resp. left), see [16].
2. $A^*$ factors on the left (resp. right) if and only if $A$ is an unital, see [16].
3. If $A^*$ factors on the left (resp. right), then $(AA^*)^\perp \cong A^{***}$ (resp. $(A^*A)^\perp \cong A^{***}$).

**Example 2.10.**

i) Let $G$ be a locally compact group. Take $B = c_0(G)$ and $B^* = A = \ell^1(G)$. Since $\ell^1(G)\ell^1(G) = \ell^1(G)$, by proceeding theorem we conclude that $c_0(G)\ell^1(G) = c_0(G)$.

ii) Let $\omega : G \to \mathbb{R}^+ \setminus \{0\}$ be a continuous function on a locally compact group $G$.

Suppose that $L^1(G,\omega) = \{f \ is \ Borel \ measurable : \|f\|_\omega = \int_G |f(s)\omega(s)|ds < \infty\}, see [6].$



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Let $X$ be a closed separable subalgebra of $L^1(G,\omega)$. Then by using [6, Theorem 7.1] and [24, Lemma 3.2], $X$ is $WSC$ and it has a sequential $BAI$, respectively. Therefore by above results if $X^*X = X^*$, then $X$ is unital and $(XX^*)^\perp = \{0\}$. On the other hand, if the identity of $G$ has a countable neighborhood base, then $L^1(G,\omega)$ has a sequential $BAI$. Now let $L^1(G,\omega)^* = L^\infty(G,\frac{1}{\omega})$ factors on the left. Hence

$$LUC(G,\frac{1}{\omega}) = L^\infty(G,\frac{1}{\omega})L^1(G,\omega) = L^\infty(G,\frac{1}{\omega}),$$

Where $L^\infty(G,\frac{1}{\omega}) = \{f \text{ is Borel measurable}: \|f\|_{\infty,\omega} = ess\ sup_{s\in G}\frac{|f(s)|}{\omega(s)} < \infty\}$.

Consequently by above results, $L^1(G,\omega)$ is unital and $RUC(G,\frac{1}{\omega})^\perp = \{0\}$, where

$$RUC(G,\frac{1}{\omega}) = L^1(G,\omega)L^\infty(G,\frac{1}{\omega}).$$

Thus we have $L^1(G,\omega)^{**} = RUC(G,\frac{1}{\omega})^*$.

**Theorem 2.11.** Suppose that $B$ is a left Banach $A-module$ and it has a $WLBAI$ $(e_\alpha)_\alpha \subseteq A$. Then we have the following assertions.

1) $B$ factors on the left.
2) If $A^*$ factors on the left, then $B^*$ factors on the left.

*Proof.*

1) By using Lemma 2.8, the result follows.
2) Let $b'' \in B^{**}$ and $b' \in B^*$. Since $\pi_\ell^{**}(b'',b') \in A^*$ and $A^*$ factors on the right, there are $a' \in A^*$ and $a \in A$ such that $\pi_\ell^{**}(b'',b') = a'a$. Without loss generality, we let $e_\alpha \xrightarrow{w^*} e''$ where $e''$ left unit for $A^{**}$. Then for every $b \in B$, we have

$$\langle \pi_\ell^{****}(b',e''),b\rangle = \langle b',\pi_\ell^{***}(e'',b)\rangle = \lim_\alpha \langle b',\pi_\ell(e_\alpha b)\rangle = \langle b',b\rangle.$$

It follows that $\pi_\ell^{****}(b',e'') = b'$. Now, we have the following equality

$$\begin{aligned}\langle b'',\pi_\ell^*(b',e_\alpha)-b'\rangle &= \langle b'',\pi_\ell^{****}(b',(e_\alpha - e''))\rangle \\ &= \langle \pi_\ell^{*****}(b'',b'),(e_\alpha - e'')\rangle = \langle \pi_\ell^{**}(b'',b'),(e_\alpha - e'')\rangle \\ &= \langle a'a,(e_\alpha - e'')\rangle = \langle a', ae_\alpha - ae''\rangle \\ &= \langle a', ae_\alpha - a\rangle \to 0.\end{aligned}$$

It follows that $\pi_\ell^*(b',e_\alpha) \xrightarrow{w} b'$, and so by *Cohen's factorization*, we are done.



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**Theorem 2.12.** Suppose that $B$ is a right Banach $A-module$ and it has a *RBAI* $(e_\alpha)_\alpha \subseteq A$. Then we have the following assertions.

1) $B$ factors on the right.
2) Let $Z^{\ell}_{A^{**}}(B^{**}) = B^{**}$. If $A^*$ factors on the right, then $B^*$ factors on the right.

*Proof.*

1) By using Lemma 2.8, the result follows.
2) Let $b'' \in B^{**}$ and $b' \in B^*$. First we show that $\pi_r^{****}(b', b'') \in A^*$. Suppose that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. Since $Z^{\ell}_{A^{**}}(B^{**}) = B^{**}$, for each $b'' \in B^{**}$, we have $\pi_r^{***}(b'', a''_\alpha) \xrightarrow{w^*} \pi_r^{***}(b'', a'')$. Then

$$\langle \pi_r^{****}(b', b''), a''_\alpha \rangle = \langle \pi_r^{***}(b'', a''_\alpha), b' \rangle \to \langle \pi_r^{***}(b'', a''), b' \rangle = \langle \pi_r^{****}(b', b''), a'' \rangle.$$

Consequently $\pi_r^{****}(b', b'') \in (A^{**}, weak^*)^* = A^*$. Since $A^*$ factors on the right, there are $a' \in A^*$ and $a \in A$ such that $\pi_r^{****}(b', b'') = a'a$. Without loss generality, we let $e_\alpha \xrightarrow{w^*} e''$ where $e''$ right unit for $A^{**}$. Then for each $b \in B$, we have

$$\langle \pi_r^{**}(e'', b'), b \rangle = \langle b', \pi_r^{***}(b, e'') \rangle = \lim_\alpha \langle b', \pi_r(b, e_\alpha) \rangle = \langle b', b \rangle.$$

It follows that $\pi_r^{**}(e'', b') = b'$. Now we have the following equality

$$\begin{aligned} \langle b'', \pi_r^{**}(e_\alpha, b') - b' \rangle &= \langle b'', \pi_r^{**}(e_\alpha, b') - \pi_r^{**}(e'', b') \rangle \\ &= \langle \pi_r^{****}(b', b''), e_\alpha - e'' \rangle = \langle a'a, e_\alpha - e'' \rangle \\ &= \langle a', ae_\alpha - ae'' \rangle = \langle a', ae_\alpha - a \rangle \to 0. \end{aligned}$$

Thus $\pi_r^{**}(e_\alpha, b') \xrightarrow{w} b'$. Consequently by *Cohen's* factorization, we are done.

**Corollary 2.13.** Suppose that $B$ is a Banach $A-bimodule$ and it has a *BAI* $(e_\alpha)_\alpha \subseteq A$. Then we have the following assertions.

1) $B$ factors.
2) Let $Z^{\ell}_{A^{**}}(B^{**}) = B^{**}$. If $A^*$ factors on the both side, then $B^*$ factors on the both side.

**Example 2.14.** Assume that $G$ is a locally compact group. We know that $L^1(G)$ is a $M(G)-bimodule$. Since $M(G)L^1(G) \neq M(G)$ and $L^1(G)M(G) \neq M(G)$, by using Theorem 2.11 and 2.12, we conclude that every *LBAI* or *RBAI* for $L^1(G)$ is not *LBAI*



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or $RBAI$ for $M(G)$, respectively.

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