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Solving Fredholm Integral Equations with Bernstein Multi-Scaling Functions

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Abstract

In this article, efficient numerical methods for finding solution of linear and nonlinear Fredholm integral equations of the second kind based on Bernstein multi scaling functions are presented. Initially, the properties of these functions, which are a combination of block-pulse functions on $[0,1)$, and Bernstein polynomials with the dual operational matrix are presented. Then these properties are used for the purpose of conversion of the mentioned integral equation to a matrix equation which is compatible to an algebraic equations system. The imperative of the Bernstein multi scaling functions for proper quantitative values of m and k, have a high accuracy and specifically the relative errors of the numerical solutions will be minimum. The presented methods from the computational viewpoint are very simple and attractive and the numerical examples at the end show the efficiency and accuracy of these methods.

Introduction

The computational approach for solution of integral equations is an essential branch of the scientific inquiry. In recent years many different basic functions such as orthogonal functions and wavelets for the estimation of the solution to linear and nonlinear integral equations have been used. The orthogonal functions are categorized into four groups [1] that are as follow:

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i) The sets of piecewise constant orthogonal functions (e.g., block-pulse, Walsh, Haar, etc.),

ii) The sets of orthogonal polynomials (e.g., Legendre, Chebyshev, Laguerre, etc.),

iii) The sets of sine-cosine functions in the Fourier series,

iv) The sets of hybrid functions which often are composed of orthogonal polynomials mentioned in (ii) and (iii) with the block-pulse functions on $[0,1)$ (e.g., hybrid Legendre, hybrid Chebyshev, hybrid Fourier, etc.).

These hybrid functions have recently been used as a very powerful and useful mathematical tools for the solutions of integral equations. The primly work in system of analysis with the hybrid functions [2] refers to [3], [4].

In this article, firstly we introduce the Bernstein multi-scaling functions (BMS functions) and dual operational matrix of them. Then we used these functions which make a non orthogonal system, to approximate the solution of linear and nonlinear Fredholm integral equations of the second kind in the form

$$y(t) = g(t) + \lambda \int_0^1 \kappa(t,s)\, y(s) ds, \qquad 0 \le t \le 1, \tag{1}$$

and

$$y(t) = g(t) + \lambda \int_0^1 \kappa(t,s) h(s, y(s)) ds, \qquad 0 \le t \le 1, \tag{2}$$

where $g(t) \in L^2[0,1)$ and $\kappa(t,s) \in L^2([0,1)\times[0,1))$ are known analytic functions, λ is the suitable constant, $y(t)$ is the unknown function to be determined and $h(s, y(s))$ is nonlinear in $y(s)$.

We presume that Eqs. (1) and (2) have a unique solution $y(t)$ that will be determined. The methods consist of expanding the solution by BMS functions with unknown coefficients. The main characteristic of this technique is that it reduces these equations to those of an easily solvable system of algebraic equations, thus greatly simplifying the problem. Various computational techniques for solving the equations (1) and (2) have been developed in the literature (see for example [5-19] and the references therein). In this section, some of them will be presented. Authors [5] suggested rationalized Haar wavelets for solving linear Fredholm integral equations. Babolian et al. [6] introduced triangular orthogonal functions for solving these equations and the

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same equations have been solved by the Laguerre series method and Legendre series method in [7] and [8], respectively. The hybrid functions are also used in the literature, such as [2], [9-12].

In [13] the nonlinear Volterra-Fredholm-Hammerstein integral equations are solved by using the Legendre wavelets. Ordokhani in [14] applied rationalized Haar functions for solving these equations. Chebyshev approximation method for solving non linear integral equations of Hammerstein type was introduced in [15] and Walsh hybrid function method for solving the Fredholm-Hammerstein integral equations was presented in [16]. Haar wavelets and periodic harmonic wavelets are also applied for solving these problems in [17-19].

From the imperative advantages of BMS functions one can refer to high accuracy of approximate solutions with less computational costs for the quantitative values of m and k compare to other hybrid functions [2], [9], [10].

Also in this article we compare the results of presented method with the result of [2], [9], [10] which shows superior method with respect to these methods in tables (1-4) .

The organization of this article is as follows: in Section 2 we give some basic definitions and in Section 3, we introduce the BMS functions and dual operational matrix of them. Section 4 is devoted to the function approximation by using Bernstein multi-scaling functions. Section 5 is devoted to the solution of linear Fredholm integral equations of the second kind. Solution of nonlinear Fredholm integral equations of the second kind will be derived in Section 6. In section 7, we provide some numerical examples. The final Section offers our conclusion.

Basic definitions

Definition : For $m \geq 0$, the Bernstein polynomials (B-polynomials) of m -th degree are defined on the interval $[0,1]$ as [20]

$$B_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i}, \qquad i = 0,1,\ldots,m,$$

where

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$$\binom{m}{i} = \frac{m!}{i!(m-i)!}.$$

There are $m+1$, m-th degree B-polynomials. For mathematical convenience, we usually set, $B_{i,m}(t) = 0$, if $i < 0$ or $i > m$.

$\{B_{i,m}(t), i = 0,1,\ldots,m\}$ in Hilbert space $L^2[0,1]$, is a complete non orthogonal set [21].

Definition : For any positive integer m, a set of block-pulse functions is defined on the interval $[0,1)$ as [22]

$$b_i(t) = \begin{cases} 1, & \frac{i}{m} \le t < \frac{i+1}{m}, \\ 0, & otherewise, \end{cases}$$

where $i = 0,1,\ldots,m-1$.

There are some properties for these functions. The most important properties are disjointness, orthogonality and completeness. The disjointness property can be clearly obtained from the definition of these functions as

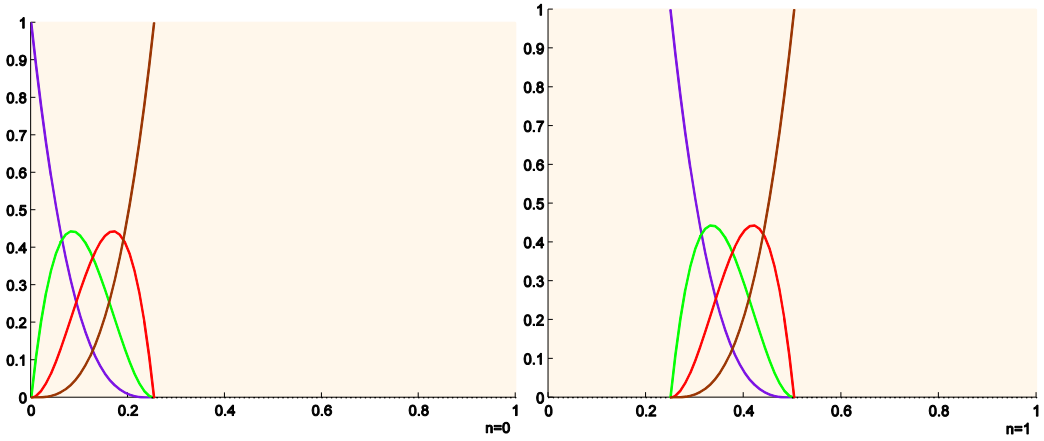
$$b_i(t)b_j(t) = \begin{cases} b_i(t), & i = j, \\ & i, j = 0,1,\ldots,m-1. \\ 0, & i \ne j, \end{cases}$$

The orthogonality of these functions is expressed by the relation

$$\int_0^1 b_i(t)b_j(t)dt = \frac{1}{m}\delta_{ij},$$

where δ_{ij} is the Kroneker delta.

Some properties of BMS functions



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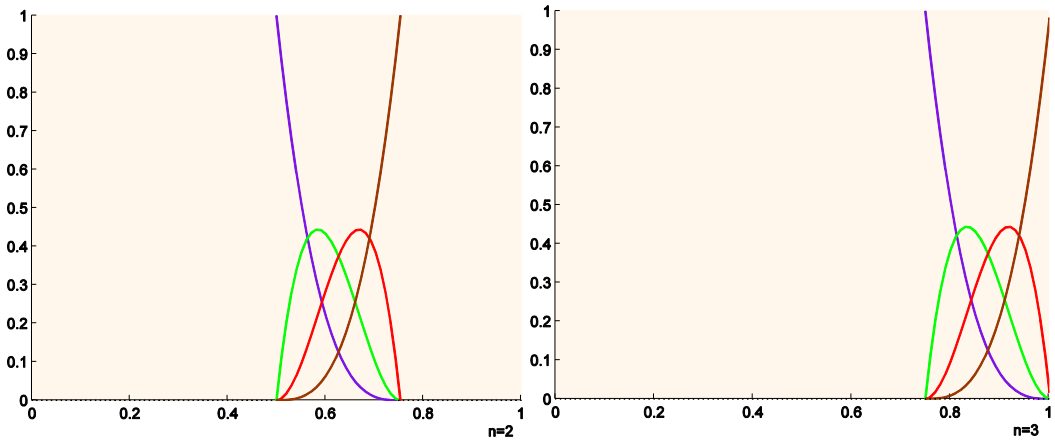


Figure 1: Graph of BMS functions for (*m* = 3*; k* = 4)

1. Definition of BMS functions

For $m \geq 1$ and any positive integer $k > 1$, the BMS functions $\psi_{i,n}(t), i = 0,1,\ldots,m$ and $n = 0,1,\ldots,k-1$ are defined on the interval $[0,1)$ as

$$\psi_{i,n}(t) = \begin{cases} B_{i,m}(kt-n), & \frac{n}{k} \leq t < \frac{n+1}{k}, \\ 0, & otherewise, \end{cases}$$

where m is the degree of B-polynomials on the interval $[0,1]$, n is the translation argument and t is the normalized time. The graph of BMS functions for $(m = 3, k = 4)$ is plotted in Figure 1.

2. Dual operational matrix

If

$$\phi(t) = \left[\psi_{0,0}, \psi_{1,0}, \ldots, \psi_{m-1,0}, \psi_{m,0}, \ldots, \psi_{0,k-1}, \psi_{1,k-1}, \ldots, \psi_{m-1,k-1}, \psi_{m,k-1}\right]^T (t),$$

be a vector function of BMS functions on the interval $[0,1)$, then with taking integration of the cross product of two of these vector functions, a matrix of $k(m+1) \times k(m+1)$ dimensional will be resulted which will be indicated as:

$$D = <\phi, \phi> = \int_0^1 \phi(t)\phi^T(t)dt. \qquad (3)$$

This matrix is known by dual operational matrix of $\phi(t)$ and will be calculated as

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$$D=\frac{1}{k}\begin{bmatrix} Q & \bar{0} & \cdots & \bar{0} \\ \bar{0} & Q & \cdots & \bar{0} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{0} & \bar{0} & \cdots & Q \end{bmatrix},$$

where $\bar{0}$ is the zero matrix $(m+1)\times(m+1)$ and Q is the dual operational matrix of B-polynomials on the interval $[0,1]$ given by [23] as:

$$Q=\frac{1}{2m+1}\begin{bmatrix} \frac{\binom{m}{0}\binom{m}{0}}{\binom{2m}{0}} & \frac{\binom{m}{0}\binom{m}{1}}{\binom{2m}{1}} & \cdots & \frac{\binom{m}{0}\binom{m}{m}}{\binom{2m}{m}} \\ \frac{\binom{m}{1}\binom{m}{0}}{\binom{2m}{1}} & \frac{\binom{m}{1}\binom{m}{1}}{\binom{2m}{2}} & \cdots & \frac{\binom{m}{1}\binom{m}{m}}{\binom{2m}{m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\binom{m}{m}\binom{m}{0}}{\binom{2m}{m}} & \frac{\binom{m}{m}\binom{m}{1}}{\binom{2m}{m+1}} & \cdots & \frac{\binom{m}{m}\binom{m}{m}}{\binom{2m}{2m}} \end{bmatrix}.$$

As you observe, Q is an invertible matrix.

Function approximation

A function $f(t)$ defined over $[0,1]$ may be expanded in terms of BMS functions as

$$y(t)\approx\sum_{n=0}^{k-1}\sum_{i=0}^{m}c_{i,n}\,\psi_{i,n}(t)=C^T\phi(t),$$

where $\phi(t)$ is the vector function defined before and C is a $k(m+1)\times 1$ vector given by

$$C^T=\left[c_{0,0},c_{1,0},\ldots,c_{m-1,0},c_{m,0},\ldots,c_{0,k-1},c_{1,k-1},\ldots,c_{m-1,k-1},c_{m,k-1}\right],$$

and can be obtained

$$C^T=\left(\int_0^1 f(t)\phi^T(t)dt\right)D^{-1}. \tag{4}$$

We can also approximate the function $\kappa(t,s)\in L^2([0,1)\times[0,1))$ as follows:

$$\kappa(t,s)\approx\phi^T(t)K\,\phi(s),$$

where K is a $k(m+1)\times k(m+1)$ matrix and can be calculated as

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$$K=\left[K_{0,0},K_{1,0},\ldots,K_{m,0},\ldots,K_{0,k-1},K_{1,k-1},\ldots,K_{m,k-1}\right], \tag{5}$$

and $\left\{K_{i,n}(t)\right\}_{i=0,\,n=0}^{m,\,k-1}$ are k $(m+1)\times 1$ column vectors and to calculate them firstly, we approximate $\kappa(t,s)$ in terms of $\left\{\psi_{i,n}(s)\right\}_{i=0,\,n=0}^{m,\,k-1}$ as

$$\kappa(t,s)\approx \xi^{T}(t)\,\phi(s),$$

where

$$\xi(t)=\left[\xi_{0,0},\,\xi_{1,0},\ldots,\,\xi_{m-1,0},\,\xi_{m,0},\ldots,\,\xi_{0,k-1},\,\xi_{1,k-1},\ldots,\,\xi_{m-1,k-1},\,\xi_{m,k-1}\right]^{T}(t),$$

and using Eq. (4) we can obtain the elements of vector $\xi(t)$ for $i=0,1,\ldots,m, n=0,1,\ldots,k-1$. Now, we approximate all functions $\left\{\xi_{i,n}(t)\right\}_{i=0,\,n=0}^{m,\,k-1}$ in terms of $\psi_{i,n}(t)$ for $i=0,1,\ldots,m, n=0,1,\ldots,k-1$ as

$$\xi_{i,n}(t)\approx\sum_{n=0}^{k-1}\sum_{i=0}^{m}k_{i,n}\,\psi_{i,n}(t)=K_{i,n}^{T}\,\phi(t), \tag{6}$$

where using Eq. (4), $\left\{K_{i,n}(t)\right\}_{i=0,\,n=0}^{m,\,k-1}$ can be obtained from Eq. (6).

Solution of linear Fredholm integral equations of the second kind

Consider the linear Fredholm integral equation of the second kind given in Eq. (1).

Firstly, we approximate $y(t)$ with BMS functions as

$$y(t)\approx C^{T}\phi(t). \tag{7}$$

Likewise, $g(t)$ and $\kappa(t,s)$ are also approximated with BMS functions as follows:

$$g(t)\approx G^{T}\phi(t),\qquad \kappa(t,s)\approx\phi^{T}(t)K\,\phi(s), \tag{8}$$

where G and K are defined similarly to Eqs. (4) and (5), respectively. Replacing Eqs. (7) and (8) into Eq. (1) we obtain

$$\phi^{T}(t)C=\phi^{T}(t)G+\lambda\int_{0}^{1}\phi^{T}(t)K\,\phi(s)\,\phi^{T}(s)C\,ds. \tag{9}$$

Using Eq. (3) we have

$$\phi^{T}(t)C=\phi^{T}(t)G+\lambda\phi^{T}(t)K\,DC. \tag{10}$$

Therefore we get

$$C=G+\lambda K\,DC, \tag{11}$$

and so by rewriting Eq. (11) we will have

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$$\left(I - \lambda K D\right)C = G, \tag{12}$$

where I is the $k(m+1)\times k(m+1)$ identity matrix and Eq. (12) is a system of linear algebraic equations which can be solved for C . Once C is known, $y(t)$ can be calculated from Eq. (7).

Solution of nonlinear Fredholm integral equations of the second kind

Consider the non-linear Fredholm integral equation of the second kind given in Eq. (2).

Let

$$z(t) = h(t, y(t)), \qquad 0 \le t \le 1. \tag{13}$$

Using Eq. (2) we have

$$z(t) = h(t, g(t) + \lambda \int_0^1 \kappa(t,s) z(s) ds). \tag{14}$$

Now, we approximate $z(t)$ and $\kappa(t,s)$ with BMS functions as

$$z(t) \approx C^T \phi(t), \qquad \kappa(t,s) \approx \phi^T(t) K \phi(s), \tag{15}$$

where C and K are defined same as Eqs. (4) and (5), respectively. By applying Eqs. (15) and (3) we can write the integral part of Eq. (14) as

$$\int_0^1 \kappa(t,s) z(s) ds = \int_0^1 \phi^T(t) K \phi(s) \phi^T(s) C \, ds = \phi^T(t) K D C. \tag{16}$$

Replacing Eq. (16) into Eq. (14) we obtain

$$z(t) = h(t, g(t) + \lambda \phi^T(t) K D C). \tag{17}$$

In order to construct the approximation for $z(t)$ we collocate Eq. (17) in $k(m+1)$ points. Suitable collocation points are Newton-Cotes nodes as [24]

$$t_p = \frac{2p-1}{2k(m+1)}, \qquad p = 1, 2, \ldots, k(m+1). \tag{18}$$

So, we have Eq. (17) as

$$z(t_p) = h(t_p, g(t_p) + \lambda \phi^T(t_p) K D C), \qquad p = 1, 2, \ldots, k(m+1). \tag{19}$$

Eq. (19) is a system of nonlinear algebraic equations which can be solved for the elements of C using Newton's iterative method.

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Now, with substituting Eq. (13) into Eq. (2) we have

$$y(t) = g(t) + \lambda \int_0^1 \kappa(t,s) z(s) ds, \qquad 0 \le t \le 1. \tag{20}$$

And therefore by replacing Eq. (16) into Eq. (20) we obtain the approximation solution of Eq. (2) as

$$y(t) = g(t) + \lambda \, \phi^T(t) K D C. \tag{21}$$

Illustrative Examples

In this section, we apply the method presented in this article and solve six examples. The computations associated with the examples were performed using Matlab 7.1.

Table1: Numerical results for Example 1.

k	m	$\|y_{exact} - y_{approx}\|_2$		$cond(I - \lambda K D)$	
		Presented method	Method of [9]	Presented method	Method of [9]
2	2	6.9204E-004	1.253E-001	1.6794	2
4	2	8.7592E-005	1.586E-002	1.6262	2
8	2	1.0984E-005	1.989E-003	1.6135	2
16	2	1.3740E-006	2.488E-004	1.6103	2
2	3	2.1780E-005	3.845E-003	1.6540	2
4	3	1.3792E-006	4.333E-004	1.6204	2
8	3	8.6484E-008	9.909E-005	1.6121	2

Example 1: Consider the linear Fredholm integral equation given in [9] by

$$y(t) = (1-t)e^t + t + \int_0^1 t^2 e^{s(t-1)} y(s) ds. \tag{22}$$

The exact solution is $y(t) = e^t$. For this example we consider the L^2-norm of errors and condition numbers of systems in norm 2 which can be shown by

$$\|y_{exact} - y_{approx}\|_2 = \left(\int_0^1 \left(y_{exact} - y_{approx} \right)^2 dt \right)^{\frac{1}{2}},$$

and

$$Cond(M) = Cond(M, 2) = \|M\|_2 \left\|M^{-1}\right\|_2.$$

These values for equal basis functions for the BMS functions and hybrid Legendre functions [9] in Table 1 are compared. As you observe in this table, the BMS functions have more accuracy compared with the hybrid Legendre functions. In addition,

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condition numbers of system for two of these functions are approximately equal to each others.

Table 2: Numerical results for Example 2.

k	m	$\|y_{exact} - y_{approx}\|_2$		$cond(I - \lambda K D)$	
		Presented method	Method of [9]	Presented method	Method of [9]
2	2	8.5140E-002	1.541E+001	3.6413	3
4	2	8.3820E-003	1.517E+000	2.1418	3
8	2	1.0711E-003	1.939E-001	1.7252	2
16	2	1.3463E-004	2.437E-002	1.6593	2
2	3	2.7736E-003	5.021E-001	3.8765	3
4	3	8.3310E-004	1.508E-001	1.9929	2
8	3	5.3053E-005	9.633E-003	1.6924	2

Example 2: Consider the linear Fredholm integral equation given in [9] by

$$y(t) = \cos(2\pi t) + \frac{1}{2}\sin(4\pi t) - \int_0^1 \sin(4\pi t + 2\pi s)\, y(s)\, ds. \tag{23}$$

The exact solution is $y(t) = \cos(2\pi t)$. For this example we also consider the L^2-norm of errors beside condition numbers of system in norm 2. In Table 2, the results obtained by the BMS functions are compared with the results of the hybrid Legendre functions [9]. As we see from this table, it is clear that the result obtained by the present method is superior to that by [9]. In addition, condition numbers of system for two of these functions are approximately equal to each others.

Example 3: Consider the linear Fredholm integral equation given in [10] by

$$y(t) = e^{2t+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2t-\frac{5}{3}s}\, y(s)\, ds. \tag{24}$$

The exact solution is $y(t) = e^{2t}$. For this example we consider the norm infinity of errors and condition numbers of system in norm infinity which can be shown by

$$\|y_{exact} - y_{approx}\|_\infty = \max_{t\in[0,1]} |y_{exact}(t) - y_{approx}(t)|,$$

and

$$Cond(M) = Cond(M, \infty) = \|M\|_\infty \|M^{-1}\|_\infty,$$

and are tabulated these values for equal basis functions for the BMS functions and hybrid Taylor functions [10] in Table 3. The advantage of presented method compared with the method of [10] is obvious, because by the same number of basis functions,

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norm infinity of errors and condition numbers of systems in present method are lower. Also we solved this example by using the presented method with $(m=10, k=2)$ and the approximate solutions are compared with the results of hybrid Legendre functions [2] in Table 4.

Table3: Numerical results for Example 3.

k	m	$\|y_{exact}-y_{approx}\|_\infty$		$cond(I-\lambda K D)$	
		Presented method	Method of [10]	Presented method	Method of [10]
10	2	4.3967E-004	3.0057E-002	4.0359	6.0920
20	2	5.8164E-005	8.6689E-003	4.0622	7.0734
40	2	7.1156E-006	2.3166E-003	4.0748	7.6292
80	2	8.7995E-007	5.9816E-004	4.0810	7.9248
10	3	6.3003E-006	2.8930E-002	4.0492	7.1086
20	3	4.1611E-007	7.3513E-003	4.0685	8.3168
40	3	2.5433E-008	1.8471E-003	4.0779	9.0048
80	3	1.5718E-009	4.6254E-004	4.0825	9.3718

Example 4: Consider the nonlinear Fredholm integral equation given in [25]

$$y(t)=1+t+(1-\frac{3}{2}\ln(3)+\frac{\sqrt{3}}{6}\pi)t^2+\int_0^1 2t^2 s\ln(y(s))ds,\quad 0\le t\le 1, \tag{25}$$

with the exact solution $y(t)=1+t+t^2$. This example is solved by using the method described in section 6 with $(m=6, k=2,4)$. The comparison among approximate solutions of the present method and the methods in [25] and [26] with the exact solution is shown in Table 5. As we see from this table, it is clear that the result obtained by the present method is superior to that by B-polynomials and the method in [26].

Table4: Approximate and exact solution for Example 3.

t	BMS functions For m=10, k=2	Method of [2] For m=11, n=2	Exact solution
0.0000	1.00000000000	1.00000000000	1.00000000000
0.0625	1.13314845307$	1.13314845282	1.13314845307
0.1250	1.28402541669	1.28402541666	1.28402541669
0.1875	1.45499141462	1.45499141434	1.45499141462
0.2500	1.64872127069	1.64872127063	1.64872127070
0.3125	1.86824595743	1.86824595710	1.86824595743
0.3750	2.11700001662	2.11700001648	2.11700001661
0.4375	2.39887529397	2.39887529358	2.39887529397
0.5000	2.71828182847	2.71828182826	2.71828182846
0.5625	3.08021684892	3.08021684845	3.08021684892
0.6250	3.49034295747	3.49034295718	3.49034295746
0.6875	3.95507672292	3.95507672235	3.95507672292
0.7500	4.48168907032	4.48168906994	4.48168907034
0.8125	5.07841903717	5.07841903648	5.07841903718
0.8750	5.75460267601	5.75460267546	5.75460267601
0.9375	6.52081912034	6.52081911947	6.52081912033

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Table5: Numerical results for Example 4.

t	Presented method		Method of [25] for m=6	Method of [26] for N=6	Exact solution
	m=6, k=2	m=6, k=4			
0.0	1.0000000000000	1.0000000000000	1.0000000000000	1.000000	1
0.1	1.1099999998945	1.1099999999996	1.1099999949939	-	1.11
0.2	1.2399999995780	1.2399999999982	1.2399999799756	1.238432	1.24
0.3	1.3899999990506	1.3899999999961	1.3899999549451	-	1.39
0.4	1.5599999983122	1.5599999999930	1.5599999199024	1.553726	1.56
0.5	1.7499999973628	1.7499999999891	1.7499998748475	-	1.75
0.6	1.9599999962024	1.9599999999842	1.9599998197805	1.945884	1.96
0.7	2.1899999948310	2.1899999999786	1.1899997547012	-	2.19
0.8	2.4399999932487	2.4399999999720	2.4399996796097	2.414905	2.44
0.9	2.7099999914554	2.7099999999646	2.7099995945060	-	2.71

Example 5: Consider the nonlinear Fredholm integral equation [18]

$$y(t) = e^{t+1} - \int_0^1 e^{t-2s} \left(y(s)\right)^3 ds, \quad 0 \le t \le 1, \tag{26}$$

where the exact solution is $y(t) = e^t$. This example is solved by using the method described in section 6 with $(m = 3, k = 2, 4)$. The comparison among approximate solutions of the present method and the method in [18] with the exact solution is shown in Table 6. As it is shown in this table, with increasing the values of m and k (specially m), the accuracy of results increased as well and also BMS functions for the less basis function have high accuracy compared with the Haar wavelets [18].

Table6: Numerical results for Example 5.

t	BMS functions m=3		Method of [18] for k=32	Exact solution
	k=2	k=4		
0.1	1.105261984	1.105179330	1.107217811	1.105170918
0.2	1.221558509	1.221409056	1.218102916	1.221402757
0.3	1.350030409	1.349865461	1.341165462	1.349858806
0.4	1.491958550	1.491836034	1.474918603	1.491824696
0.5	1.649038709	1.648739647	1.667402633	1.648721268
0.6	1.822268942	1.822132669	1.833861053	1.822118797
0.7	2.014009496	2.013763091	2.016679830	2.013752703
0.8	2.225823852	2.225551897	2.217456630	2.225540923
0.9	2.459823796	2.459621801	2.437978177	2.459603104

Example 6: Consider the nonlinear Fredholm integral equation [15]

$$y(t) = \exp(1)t + 1 - \int_0^1 (t+s)\, e^{y(s)} ds, \quad 0 \le t \le 1, \tag{27}$$

where the exact solution is $y(t) = t$. We solve this example by using the method described in section 6 with $(m = 2, k = 5, 10)$ and $(m = 4, k = 5)$. The comparison among absolute errors of the present method and method in [15] is shown in Table 7. As this table depicts, with increasing the values of m and k (specially m), the accuracy of

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results increased as well and also BMS functions have high accuracy compared with the Chebyshev polynomials [15] (with the same degree).

Table7: Numerical results for Example 6.

t	BMS functions			Method of [15]	
	m=2		m=4	N=3	N=5
	k=5	k=10	k=5		
0.0	1.1 E-006	8.3 E-008	2.8 E-009	1.0 E-006	2.3 E-005
0.2	9.7 E-007	7.7 E-008	3.4 E-009	3.2 E-004	3.1 E-005
0.4	8.2 E-007	7.0 E-008	4.0 E-009	2.6 E-004	1.7 E-005
0.6	6.6 E-007	6.3 E-008	4.6 E-009	2.1 E-004	7.5 E-006
0.8	5.0 E-007	5.6 E-008	5.2 E-009	1.8 E-004	2.1 E-005

Conclusions

In this article, the methods of approximate solution of linear and nonlinear Fredholm integral equations by utilizing BMS functions were presented. In the presented methods, dual operational matrix of these functions were used for resolving the equations. Since this matrix comprises, many zeros elements, it can bring about a numerical accurate result with high reliability of achieving the desired results and these methods are very attractive. With solving six examples, the methods were evaluated and as tables 1 to 7 show, with increasing the quantities of m and k (particularly m) the error drops to zero rapidly.

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