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A meshless Method for Solving an inverse Time-dependent Heat Source Problem

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Abstract

Inverse time-dependent heat source problems have an important role in many branches of science and technology. The aim of this paper is to solve these classes of problems using a variational iteration method(VIM). The method applied does not require discretization of the region, as in the case of classical methods based on the finite difference method, the boundary element method or the other methods. Applying this method, we obtain a stable approximation to an unknown source term in an inverse heat equation from over-specified data that the source term is only time-dependent. Some numerical examples using this approach are presented and discussed.

Introduction

The inverse problem of determining an unknown heat source function in the heat conduction equation has been considered in many theoretical papers, notably [1-3].

In this paper, we consider the inverse time-dependent heat source problem of determination a pair of functions $(u(x,t), f(t))$ in the following heat conduction equation with a time-dependent source:

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + f(t); \qquad 0 < x < 1, \quad 0 < t < t_{\max}, \tag{1}$$

with the initial - boundary conditions:

$$u(x,0) = \phi(x); \qquad 0 \le x \le 1, \tag{2}$$

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$$u(0,t)=\mu_1(t), \qquad u(1,t)=\mu_2(t); \qquad 0\le t\le t_{\max}, \tag{3}$$

Subject to the over-specified condition:

$$u(x^*,t)=E(t); \quad 0\le t<t_{\max}, \quad E(t)\neq 0, \tag{4}$$

where $x^*\in(0,1)$ is the interior location of a thermocouple recording the temperature measurement (4), $t_{\max}$ is the final measurement time and ϕ, μ_1, μ_2 and E are given functions satisfying the compatibility conditions:

$$\phi(x^*)=E(0), \qquad \phi(0)=\mu_1(0), \qquad \phi(1)=\mu_2(0), \tag{5}$$

$$\mu_1'(0)=\phi''(0)+E'(0)-\phi''(x^*), \qquad \mu_2'(0)=\phi''(1)+E'(0)-\phi''(x^*). \tag{6}$$

The problem given by equations (1) and (2) is called the characteristic problem, whilst the problem given by equations (1) and (3) is called the non-characteristic problem. Problems of this type include inverse problems, heat conduction processes, hydrology, material sciences and heat transfer problems. In the context of heat conduction and diffusion, when u represents temperature and concentration, the unknown $f(t)$ is interpreted as a heat and material source, respectively and in a chemical or a biochemical application, $f(t)$ may be interpreted as a reaction term [4].

Under an additional priori condition, the unique solvability of the inverse problem (1)-(6) can be obtained (see Theorem 1 in [5]; pp. 376 and in [6]; pp. 217). The existence and uniqueness of solutions to the similar problem has been studied in Refs. [3] and [5], but it is ill-posed since the solution does not depend continuously on the input data. [5; pp. 376], [6; pp. 217]

Several numerical methods have been proposed for solving the inverse source problem (1)-(6), for example, in [4], [5], [6] this problem is solved by the finite-difference, the boundary-element method and the fundamental solutions method, respectively. In this work, we use variational iteration method to solve this inverse source problem. The variational iteration method was proposed originally by He [7-10]. The method has been applied to a wide range of parabolic problems, see for example [16, 17].

This paper is organized in the following way:

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In the next section, we introduce basic ideas of the VIM which will be studied in this work. The application of variational iteration method is introduced in Section 3. In Section 4, the described technique is applied on several test problems to show the efficiency of the proposed approach. Some conclusions are drawn in the last section.

Basic ideas of the VIM

The VIM is used for solving a wide range of general non-linear differential equations of the form:[8, 9, 10]

$$Lu(t)+Nu(t)=F(t), \tag{7a}$$

where L and N are linear and non-linear operators, respectively, $F(t)$ is a known analytical inhomogeneous term, and u is an unknown function. According to VIM, we can construct a correction functional as follows:[9,13-19]

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(\xi,t)\{Lu_n(\xi)+N\tilde{u}_n(\xi)-F(\xi)\}d\xi; \quad n\geq 0, \tag{7b}$$

where λ is a general Lagrange multiplier [9, 11], optimally determined by using the variational theory [8, 9, 12], the index n denotes the n-th order approximation and $\tilde{u}_n$ is a restricted variation, i.e. $\delta\tilde{u}_n=0$ [8,12,18,19]. Now, we need to determine the Lagrangian multiplier λ. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. Then, the solution of the differential equation is considered as the fixed point of the following functional under the suitable choice of the initial term $u_0(t)$: [9,13-19]

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(\xi,t)\{Lu_n(\xi)+Nu_n(\xi)-F(\xi)\}d\xi; \quad n\geq 0. \tag{7c}$$

Consequently, the exact solution may be obtained by using the Banach's fixed point Theorem[13, 14]:

$$u(t)=\lim_{n\to\infty} u_n(t),$$

because according to the above theorem, for the nonlinear mapping:

$$A[u]=u(t)+\int_0^t \lambda(\xi,t)\{Lu(\xi)+Nu(\xi)-F(\xi)\}d\xi; \quad |A[u]-A[v]|\leq \gamma|u-v|, 0<\gamma<1, \forall u,v\in \mathrm{X},$$

a sufficient condition for the convergence of the VIM is strictly contraction of $A:\mathrm{X}\to\mathrm{X}$ where X is a Banach's space. Futhermore, the sequence (7c); i.e.

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$u_{n+1}(t)=A[u_n(t)]$ converges to the fixed point of A, with an arbitrary choice of $u_0 \in \mathrm{X}$, which is also the solution of the nonlinear equation (1). (see Theorem 2.3 in [13], pp. 123 and Theorem 1 in [14], pp.2529.)

Application of Variational iteration method

To obtain a partial differential equation containing only one unknown function, if the functions pair (u,f) solves the inverse problem (1) - (6), then from (1) and (4), we have:

$$E'(t)=\frac{\partial^2}{\partial x^2}u(x^*,t)+f(t); \qquad 0\le t\le t_{\max},$$

and it follows that:

$$f(t)=E'(t)-\frac{\partial^2}{\partial x^2}u(x^*,t); \qquad 0\le t\le t_{\max}. \tag{7}$$

Then the inverse problem (1)-(4) is equivalent to the following initial - boundary value problem:

$$\frac{\partial}{\partial t}u(x,t)=\frac{\partial^2}{\partial x^2}u(x,t)+E'(t)-\frac{\partial^2}{\partial x^2}u(x^*,t); \qquad 0<x<1, \quad 0<t<t_{\max}, \tag{8}$$

$$u(x,0)=\phi(x); \qquad 0\le x\le 1, \tag{9}$$

$$u(0,t)=\mu_1(t), \qquad u(1,t)=\mu_2(t); \qquad 0\le t\le t_{\max}, \tag{10}$$

which shows that equation (8) has only one unknown function $u(x,t)$. Thus equation (8) which is equivalent to equation (1), has suitable form to apply the variational iteration method.

According to He's classical variational iteration scheme and (7a), we can construct a correction functional concerning equation (8) in t -direction as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(t,s)\left(\frac{\partial}{\partial t}u_n(x,s)-\frac{\partial^2}{\partial x^2}\tilde{u}_n(x,s)-E'(s)+\frac{\partial^2}{\partial x^2}\tilde{u}_n(x^*,s)\right)ds=0; \quad n\ge 0. \tag{11}$$

To find the optimal value of λ, from (11), we have:

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(t,s)\left(\frac{\partial}{\partial t}u_n(x,s)-\frac{\partial^2}{\partial x^2}\tilde{u}_n(x,s)-E'(s)+\frac{\partial^2}{\partial x^2}\tilde{u}_n(x^*,s)\right)ds=0,$$

which after some calculation results:

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$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(t,s)\delta u_n(x,s)|_{s=t} - \int_0^t \lambda'(t,s)\delta u_n(x,s)ds$$

$$= (1+\lambda(t,s))|_{s=t}\ \delta u_n(x,t) - \int_0^t \lambda'(t,s)\delta u_n(x,s)ds = 0.$$

Hence, we obtain the following stationary conditions:

$$\lambda'(t,s) = 0,$$
$$1+\lambda(t,s)|_{s=t} = 0.$$

This in turn gives:

$$\lambda(t,s) = -1.$$

Substituting this value of the Lagrange multiplier into function (11) and from (7c) gives the iteration formula:

(12)

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (\frac{\partial}{\partial t}u_n(x,s) - \frac{\partial^2}{\partial x^2}u_n(x,s) - E''(s) + \frac{\partial^2}{\partial x^2}u_n(x^*,s))ds = 0;\ n \geq 0,$$

where u_0 may be selected as any function that just satisfies, at least, the initial or boundary conditions [7-12] but according to the Adomian's decomposition method (ADM) in t -direction which is equivalent to the VIM in t -direction [15], we assume $Lu_0(x,t) = 0$ or $u_0(x,t) = \phi(x)$, for simplicity, as the initial approximation[9]. Therefore using (12), according to the Banach's fixed point Theorem, we can find the solution of the problem (8)-(10) as a convergent sequence [13, 14]. Also, we can consider u_n as an approximation of the exact solution for sufficiently large values of n. Consequently, from the solution of the problem (8)-(10), we can obtain the approximate solution $f(t)$ by equation (7).

Illustrative examples

In this section, we present and discuss the numerical results by employing variational iteration method for some test examples. Examples have been chosen so that their analytical solutions exist. The application of the solution these examples are demonstrated in [4]. The results demonstrate that the present method is remarkably effective.

Example 3.1 . Problem (1) - (4) with assumption $\phi(x) = x^2$, $t_{\max} = 1$, $\mu_1(t) = 2t + \cos(k\pi t); k \in N$, $\mu_2(t) = 1 + 2t + \sin(k\pi t)$, $x^* = 0.5$ and

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$E(t)=0.25+2t+\sin(k\pi t)$ has the exact solution $u(x,t)=x^2+2t+\sin(k\pi t)$ and $f(t)=k\pi\cos(k\pi t)$.

Using (12), we obtain the recurrence relations:

$u_0(x,t)=\phi(x)=x^2,$

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t(\frac{\partial}{\partial t}u_n(x,s)-\frac{\partial^2}{\partial x^2}u_n(x,s)-2-k\pi\cos(k\pi t)+\frac{\partial^2}{\partial x^2}u_n(0.5,s))ds;\quad n\geq 0.$$

By the above iteration formula and after some simplifications, we obtain the following successive approximations:

$$u_0(x,t)=\phi(x)=x^2,$$
$$u_1(x,t)=x^2+2t+\sin(k\pi t),$$
$$u_2(x,t)=u_1(x,t),$$
$$\vdots$$

Finally, $u_n(x,t)=x^2+2t+\sin(k\pi t)$. Then we can write:

$$u(x,t)=u_1(x,t)=x^2+2t+\sin(k\pi t), \tag{13}$$

that we obtain exact solution after 2 iteration.

Consequently, for approximating $f(t)$, from (7) and (13), we find:

$$f(t)=E'(t)-\frac{\partial^2}{\partial x^2}u(0.5,t)=2+k\pi\cos(k\pi t)-2=k\pi\cos(k\pi t);\qquad 0\leq t\leq 1,$$

which is the exact solution of Example 3.1. This example is chosen from [4; Example 2, pp. 972, k=4], [5; Example 4.1, pp. 383, k=4] and [6; Example 2, pp. 220, k=2] to indicate that the present method is remarkably effective.

Example3.2. Problem (1) - (4) with assumption $\phi(x)=\sin(x)+\frac{1}{4}x^4$, $t_{\max}=1$, $\mu_1(t)=0$, $\mu_2(t)=e^{-t}\sin(1)+3t+\frac{1}{4}$, $x^*=0.5$ and $E(t)=e^{-t}\sin(0.5)+\frac{3}{4}t+\frac{1}{64}$ has the exact solution $u(x,t)=e^{-t}\sin(x)+3tx^2+\frac{1}{4}x^4$ and $f(t)=6t$.

Using (12), we obtain the recurrence relations:

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$$u_0(x,t)=\phi(x)=\sin(x)+\frac{1}{4}x^4,$$

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t(\frac{\partial}{\partial t}u_n(x,s)-\frac{\partial^2}{\partial x^2}u_n(x,s)+e^{-s}\sin(0.5)+\frac{\partial^2}{\partial x^2}u_n(0.5,s))ds;\quad n\geq 0.$$

By the above iteration formula and after some simplifications, we obtain the following successive approximations:

$$u_0(x,t)=\sin(x)+\frac{1}{4}x^4,$$

$$u_1(x,t)=(1-t)\sin(x)+\frac{1}{4}x^4+3tx^2-e^{-t}\sin(0.5)+(1-t)\sin(0.5),$$

$$u_2(x,t)=(1-t+\frac{t^2}{2!})\sin(x)+\frac{1}{4}x^4+3tx^2-e^{-t}\sin(0.5)+(1-t+\frac{t^2}{2!})\sin(0.5),$$

$$\vdots$$

Generally, we obtain:

$$u_n(x,t)=(1-t+\frac{t^2}{2!}-+...+(-1)^n\frac{t^n}{n!})\sin(x)+\frac{1}{4}x^4+3tx^2-e^{-t}\sin(0.5)$$
$$+(1-t+\frac{t^2}{2!}-+...+(-1)^n\frac{t^n}{n!})\sin(0.5),$$

which converge to the exact solution:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=e^{-t}\sin(x)+3tx^2+\frac{1}{4}x^4. \qquad (14)$$

Hence from (7) and (14), we find:

$$f(t)=E'(t)-\frac{\partial^2}{\partial x^2}u(0.5,t)=-e^{-t}+\frac{3}{4}+e^{-t}-6t-\frac{3}{4}=-6t;\qquad 0\leq t\leq 1,$$

which is the exact solution of Example 3.2 . This example is chosen from [6; Example 1, pp. 219] to demonstrate that the present method is remarkably effective.

Example 3.3 . Problem (1) - (4) with assumption $\phi(x)=e^x+1$, $t_{\max}=1$, $\mu_1(t)=e^t+\cos(t)$, $\mu_2(t)=e^{t+1}+\cos(t)$, $x^*=0.5$ and $E(t)=e^{t+0.5}+\cos(t)$ has the exact solution $u(x,t)=e^{x+t}+\cos(t)$ and $f(t)=-\sin(t).$

Using (12), we obtain the recurrence relations:

$$u_0(x,t)=\phi(x)=e^x+1,$$

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t(\frac{\partial}{\partial t}u_n(x,s)-\frac{\partial^2}{\partial x^2}u_n(x,s)-e^{s+0.5}+\sin(s)+\frac{\partial^2}{\partial x^2}u_n(0.5,s))ds;\quad n\geq 0.$$

By the above iteration formula and after some simplifications, we obtain the following successive approximations:

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$$u_0(x,t) = e^x + 1,$$
$$u_1(x,t) = e^x(1+t) + e^{t+0.5} - e^{0.5}(1+t) + \cos(t),$$
$$u_2(x,t) = e^x(1+t+\frac{t^2}{2!}) + e^{t+0.5} - e^{0.5}(1+t+\frac{t^2}{2!}) + \cos(t).$$

Generally, we obtain:

$$u_n(x,t) = e^x(1+t+\frac{t^2}{2!}+...+\frac{t^n}{n!}) + e^{t+0.5} - e^{0.5}(1+t+\frac{t^2}{2!}+...+\frac{t^n}{n!}) + \cos(t),$$

which converge to the exact solution:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = e^x e^t + e^{0.5+t} - e^{0.5}e^t + \cos(t) = e^{x+t} + \cos(t). \qquad (15)$$

Consequently, for approximating $f(t)$, from (7) and (15), we find:

$$f(t) = E'(t) - \frac{\partial^2}{\partial x^2}u(0.5,t) = (e^{0.5+t} - \sin(t)) - e^{0.5+t} = -\sin(t); \qquad 0 \le t \le 1,$$

which is the exact solution of Example 3.3.

Conclusion

In this paper, the variational iteration method is used for finding solution pair of the inverse one - dimension time - dependent heat source equation. The variational iteration method is show to be a powerful numerical method for the solution of ill-posed problems. The simplicity of the method and the obtained results show that this method is effective, simple and easy compared with many of the other methods. It is also shown that the variational iteration method provides an exact solution for the inverse problem (1) - (4). The given numerical examples support this claim. Also, the method applied does not require discretization of the region, as in the case of classical methods based on the finite difference method, the boundary element method or the other methods.

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References

1. A. I. Prilepko, V. V. Solov'ev, "Solvability theorems and Rote's method for inverse problems for a parabolic equation", I. Diff. Equ. 23 (1988)1230-1237.

2. V. V. Solov'ev, "Solvability of the inverse problem of finding a source", using overdetermination on the upper base for a parabolic equation, I. Diff. Equ. 25(9) (1990)1114-1119.

3. E. G. Savateev, "On problems of determining the source function in a parabolic equation", J. Inv. Ill-posed problems 3(1) (1995) 83-102.

4. J. Lin, B. Wang, Z. Lin, "Determination of a source term in a heat equation", Int. J. Comput. Maths, 87(5) (2010) 969-975.

5. A. Farcas, D. Lesnic, "The boundary-element method for the determination of a heat source dependent on one variable", J. Eng. Math. 54 (2006) 375-388.

6. L. Yan, Ch. Fu, F. Yang, "The method of fundamental solutions for the inverse heat source problem", Eng. Anal. Boundary Elem. 32 (2008) 216-222.

7. J. H. He," A new approach to nonlinear partial differential equations", Commun. Nonlinear Sci. Numer, Simulation, 2 (4) (1997) 230-235.

8. J. H. He, "Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput, Methods Appl. Mech. Eng. 167 (1998) 69-73.

9. J. H. He, "Variational iteration method-a kind of non-linear analytical technique: some examples", Int. Non-linear Mech. 34 (4) (1999) 699-708.

10. J. H. He, "Approximate analytical solution for secpage flow with fractional derivatives in porous media", Comput. Methods Appl. Mech. Eng. 167 (1998) 57-68.

11. M. Inokuti, H. Sekine, T. Mura, "General use of the Lagrange multiplier in non-linear mathematical physics", in: S. Nemat-Nasser (Ed.), Variational Method in the Mechanics of Solids, Pergamon Press, Oxford (1978) 156-162.

12. B. A. Finlayson, "The Method of Weighted Residuals and Variational Principles", Academic Press, New York (1972).

13. M. Tatari and M. Dehghan, "On the convergence of He's variational iteration method", J. Comput. Appl. Math. 207 (2007) 121-128.

[Downloaded from ndea10.khu.ac.ir on 2026-11-02]

14. L. Ahmad Soltani, A. Shiradi, "A new modification of the variational iteration method", Comput. and Math. Appl.59 (2010) 2528-2535.
15. A. M. Wazwaz, "A comparison between the variational iteration method and Adomian decomposition method", J. Comput. and Appl. Math. 207 (2007) 129-139.
16. F. Geng and Y. Lin, "Application of the variational iteration method to inverse heat source problems", J. Comput. Appl. Math. 58 (2009) 2098-2102.
17. M. Dehghan, M. Tatari, "Identifying an unknown function in a parabolic equation with overpecified data via He's variational iteration method", chaos, solutions and fractals 36 (2008)157-166.
18. J. H. He, "Variational iteration method for autonomous ordinary differential systems", Appl. Math. Comput. 114 (2000) 115-123.
19. J. H. He, "Non-pertubative methods for strongly nonlinear problems", Dissertution, de-verlag im Internet GmbH, Berlin (2006).

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